GRAVITATIONAL MODELS IN 2+1 DIMENSIONS WITH TOPOLOGICAL TERMS AND THERMO-FIELD DYNAMICS OF BLACK HOLES

By

BETTINA E. KESZTHELYI

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We consider two extensions of Einstein gravity in 2+1 dimensions. First we study some consequences of duality in three dimensions. In the case of pure gravity, a dual Ansatz is shown to lead to pure gauge configurations, and in a Weyl invariant gravitational theory, duality arises as an equation of motion. Its solutions obey the Liouville equation and describe a rotating Chern-Simons fluid in a gravitational field.

Next we analyze the theory of topologically massive gravity in three spacetime dimensions, conjectured to be renormalizable by Deser and Yang, using the nonlocal regularization. The validity of this technique, however, depends on the existence of a gauge-invariant measure for the nonlocal theory. Assuming that such a measure exists, we show that the possible obstacle to renormalizability found by Deser and Yang does not appear.

Finally thermo-field dynamics is used to derive the Hawking radiation of black holes emitting massless scalar particles or spin one half fermions. We also show how to generalize this method to particle creation in spacetimes with more than two causally disconnected regions, *e.g.*, in the gravitational field of many black holes.

CHAPTER 1
INTRODUCTION

In this work we study two 2+1 dimensional gravitational models, Weyl gravity and topologically massive gravity. We study the consequences of duality in the first case and renormalizability in the second. We also consider the Hawking radiation of black holes in 3+1 dimensions.

Field theories in lower space-time dimensions have recently attracted attention as tractable toy models for realistic 3+1 dimensional systems and also in their own right for their unusual topological properties. For example, one can hope to find 2+1 dimensional models involving gravity which have better short distance behavior than those in 3+1 dimensions. The latter are known to be nonrenormalizable due to the presence of the gravitational coupling which has negative mass dimension. Indeed, one may be able to discover 2+1 dimensional theories which are renormalizable and may help one to better understand the quantum behavior of the higher dimensional theory. Classical solutions of these theories are also interesting. These can be used to study the nature of gravitational singularities or can represent physical axially symmetric solutions such as cosmic strings.

We consider two models addressing the above questions. Since they both involve gravity, first we review the main features of 2+1 dimensional pure Einstein gravity. These features are unique, but restrictive. This is why it is interesting to study its extensions as well. The properties of 2+1 gravity can

be derived from the Einstein equation which is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. Expressed in terms of the Ricci tensor we have

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} + \kappa(T_{\mu\nu} - g_{\mu\nu}T) \ , \tag{1.2}$$

where Λ is the cosmological constant and the coupling κ has mass dimension -1. The curvature tensor and the Ricci tensor both have six independent components and one can write the curvature in terms of the Ricci tensor,

$$\begin{aligned} R_{\mu\nu\lambda\rho} &= g_{\mu\lambda}R_{\nu\rho} + g_{\nu\rho}R_{\mu\lambda} - g_{\mu\rho}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\rho} \\ &- \tfrac{1}{2}R(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \ . \end{aligned} \tag{1.3}$$

From (1.2) and (1.3) one can see that the curvature tensor is completely defined by the energy-momentum tensor $T_{\mu\nu}$ and Λ. When $T_{\mu\nu} = 0$, the scalar curvature is $R = 6\Lambda$ so that any curvature effects produced by matter do not propagate through spacetime and there are no dynamical degrees of freedom. This could simply be seen by counting degrees of freedom. Depending upon the sign of Λ, the spacetime is either locally flat, de Sitter or anti-de Sitter.

Although local curvature in source-free regions is unaffected by matter, it can still produce nontrivial global effects. Even in the simplest cases such global effects are present. For example, for a point mass and $\Lambda = 0$ spacetime is flat except along the world line of the particle. In the static case, coordinates can be chosen so that the constant time surfaces are conical [1]. This conical spacetime is obtained by removing a wedge from Minkowski space and identifying points along the wedge with the wedge angle given by κm [2], where m is the particle's mass. The three dimensional geometry of a massive spinning

source is also conical, but in this case, the points that are identified across the deleted wedge differ in their time coordinate value by an amount proportional to the angular momentum of the source. Thus in the presence of a spinning source the spacetime has a "helical structure," a rotation about the source is accompanied by a shift in time [2]. This conical-helical geometry characterizes the spacetime outside of more general compact matter distributions, because matter cannot affect the local curvature in source free regions.

Despite being locally flat except along the particle's world line, these spacetimes have interesting global geometric properties. For example, there is an analogue of the Aharanov-Bohm effect in that a vector parallel transported around a loop surrounding the source experiences a nontrivial rotation, even though the loop lies entirely within flat regions of the spacetime [3]. Similarly, two geodesics passing of opposite sides of the source may intersect twice. This effect also arises in 3+1 dimensional gravity in the context of cosmic strings [4].

Here we will consider two extensions of the Einstein theory in 2+1 dimensions, Weyl gravity and topologically massive gravity (TMG). The outline of this work is as follows:

In Chapter 2 we show that Weyl gravity has self-dual solutions, and in certain cases exact solutions can be found. Our motivation for seeking such solutions, as described in Sect. 2.1 [5], is the important role they play in 3+1 dimensional theories. For example in four Euclidian dimensions self-duality of the Yang-Mills field strength leads to instanton solutions [6]. A similar condition imposed on the gravitational connection leads to the Eguchi-Hanson gravitational instantons [7].

In Sect. 2.2 we review Weyl theory in $d+1$ space-time dimensions. Then we find the most general Weyl invariant theory in 2+1 dimensions that contains

gravity, a Weyl gauge field and a real scalar field. We find some interesting features of the field equations, obtained from the Lagrangian of this theory. These are useful to predict some of the general features of the solutions. For example we show that our solutions correspond to a special type of rotating fluid immersed in a 2+1 dimensional gravitational field. This indicates that such solutions might have interesting applications in fluid mechanics. We refer back to this analogy throughout the following subsections of Chapter 2. We also show that the field equation of the Weyl gauge field in a specific gauge imposes a self-duality condition between the field strength and the gauge potential.

In Sect. 2.3 we discuss stationary solutions in this gauge. We show that they can be classified by the nonvanishing elements of the field strength tensor. We find that the case with no electric or magnetic field ($E_i = 0$, $B = 0$) reduces to Einstein gravity in flat or in de Sitter space [2]. The purely magnetic case ($E_i = 0$, $B \neq 0$) and the case with nonzero electric and magnetic field ($E_i \neq 0$, $B \neq 0$) are more interesting. In the former case we find all the static solutions, but not in the latter.

From here we proceed by solving the field equations for the $E_i = 0$, $B \neq 0$ case in Sect 2.4, and for the $E_i = 0$, $B = 0$ case in Sect 2.5. In Sect. 2.6 we continue with studying our solutions. We find that in the axisymmetric case they correspond to known 3+1 dimensional spacetimes. It is also interesting that in the proper coordinates our solutions have the same conical-helical geometry characteristic to a large class of known solutions of 2+1 gravity as mentioned above.

Weyl gravity is one of the natural generalizations of Einstein gravity if one wishes to study the quantum theory, because it has better short distance

behavior arising from conformal invariance [8]![(1)] In 2+1 dimensions there is another possible choice, topologically massive gravity [9].

In Chapter 3 we study the renormalizability of TMG by using nonlocal regularization [10]. In Sect. 3.1 we describe why it is a good candidate for a renormalizable theory with the symmetries of gravity. There are several arguments [9], including ours [10], indicating that TMG is renormalizable, but none of them can be considered a strict proof. The importance of the question is, that if these arguments are proven to be true, TMG would be the only known theory with such properties.

In Sect. 3.2 we use general power counting arguments to show why certain gravity theories are not renormalizable.

In Sect. 3.3 we review the main features of TMG, and we show that it is power counting renormalizable. However, in order to conclude that the theory is renormalizable, one also has to show that the gauge invariance may be maintained in the regulated version of the theory without giving up the desirable power counting behavior. In other words, without the use of a gauge invariant regulator, additional terms might be required to cancel the gauge transformation of the effective action, and these might spoil power-counting renormalizability. For these reasons we use gauge invariant nonlocal regularization.

In Sects. 3.4 and 3.5 we review the method of nonlocal regularization and the nonlocal Feynman rules, respectively [11].

In Sect. 3.6 we apply these rules for TMG. Unfortunately, to find a proper gauge invariant measure factor for the nonlocalized theory, or at least prove its

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(1) It is necessary to consider a generalized theory, because Einstein theory in the absence of matter has no dynamical degrees of freedom.

existence, is an extremely difficult problem. Only a perturbative method exists which allows us to calculate it to any desired order in the fields. In theories without gauge anomalies (like TMG), it is reasonable to assume its existence. We show that if this holds, the only anomaly term, found in ref. 9, does not appear, and the theory is renormalizable.

In Chapter 4 we consider another interesting problem of general relativity, namely particle creation from the vacuum in spacetimes with causally disconnected regions. It is known that the quantization of fields on such spacetimes leads to particle creation from the vacuum as a consequence of the information loss associated with the presence of the event horizon(s) [12]. In stationary spacetimes with a simply connected event horizon (such as a stationary black hole, an accelerated observer in Minkowski spacetime, or de Sitter type cosmologies), the emitted particles have a thermal spectrum [13]. This result has been first obtained by Hawking [14] for black holes. He has shown, that the effective temperature of this radiation is $T_H = \frac{\kappa}{2\pi}$, the Hawking temperature, where κ is the surface gravity of the black hole (units are chosen throughout such that $\mathrm{k} = \hbar = \mathrm{c} = G = 1$). These results have been confirmed and derived in a number of ways, and several attempts have been made to gain a better understanding of the features of the Hawking process and the physical role of the event horizon [14,15,16]. A particularly interesting approach into this direction is the one used by Israel [16], who considered the problem of particle creation on the Schwarzschild background. The idea is to quantize the fields in the full analytically extended Schwarzschild spacetime (known as the Kruskal extension) in order to keep track of particle states on the hidden side of the horizon as well. The same idea allows us to apply a quantum-statistical formalism, known as thermo-field dynamics [17].

Here we follow Israel's approach, and first we rederive the well-known results of Hawking radiation of a black hole. This method reproduces the standard results. We look for possible extensions to the case of many black holes. The organization of this chapter is as follows: In Sect. 4.1 we briefly review the canonical formulation of thermo-field dynamics for free fields [17] and show how it can be generalized to describe particle creation in general relativity [16].

In Sect. 4.2 we review Israel's paper [16]. The quantization of a massless scalar field on the Schwarzschild background is considered by using thermo-field dynamics. We conclude that the results obtained this way are equivalent to the earlier ones. In the rest of the paper we describe the possible generalizations of this approach. In Sect. 4.3 an approximate multi-black hole solution is considered as an example to demonstrate how to extend the method to space-times with many causally disconnected regions. Finally in Sect. 4.4 we derive the Hawking radiation of a black hole emitting neutrinos and antineutrinos.

CHAPTER 2
DUAL SOLUTIONS IN 2+1 DIMENSIONS

2.1 Introduction

In four Euclidean dimensions, self-duality of Yang-Mills gauge fields leads to the classical instanton solutions. A similar condition imposed on the gravitational connection leads to the Eguchi-Hanson gravitational instantons. Motivated by the recent interest in theories of lesser number of dimensions, we investigate analogs of self-duality in the classical solutions of some theories in 2+1 dimensions.

In the case of Yang-Mills gauge fields, self-duality cannot be imposed on the field strengths, as in four dimensions, but only between the gauge potential and the field strength. In order to find such a solution, we investigate a simple theory involving gravity and an Abelian gauge potential, linked by the requirement of Weyl invariance. The classical equations of motion demand duality (in a special gauge). We study stationary solutions of these equations. We first solve the equations of motion in the pure gauge case, when our theory reduces to Einstein gravity with nonzero cosmological constant. Then we find the general solution when the magnetic field does not vanish; for these the conformal factor of the two dimensional space satisfies a Liouville equation. The physical situation corresponds to a special type of fluid immersed in a 2+1 gravitational field. The fluid is a rotating perfect fluid with velocity corresponding to the normalized vector potential. Futhermore, the velocity obeys a

duality condition and a massive Klein-Gordon equation. When specializing to axisymmetric solutions, we recover either the 2+1 dimensional Gödel solution or the boundary of 4 dimensional Taub-NUT space-time solutions.

2.2 Weyl Theory in 2+1 Dimensions

In order to consider gauge fields, we use as a principle Weyl's original theory which links gravity to electromagnetism through a "gauge" principle [18]. For the reader who may be unfamiliar with this type of theory, we start with a brief review.

In the second order formalism of general relativity, the fundamental fields are the metric $g_{\mu\nu}(x)$ and the gauge field $A_\mu(x)$. In addition to the usual diffeomorphism of general relativity, Weyl requires invariance under conformal rescaling or "gauge" transformation

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \ , \tag{2.2.1}$$

$$g^{\mu\nu}(x) \to \Omega^{-2}(x) g^{\mu\nu}(x) \ , \tag{2.2.2}$$

$$A_\mu(x) \to A_\mu(x) + 2\partial_\mu \ln \Omega(x) \ . \tag{2.2.3}$$

In Einstein theory the spacetime is described by Riemannian geometry, where one has a metric connection, that is, the covariant derivative of the metric is zero,

$$\nabla_\lambda g_{\mu\nu} = 0 \ . \tag{2.2.4}$$

This implies that lengths and angles are preserved under parallel transport. In order to get a locally scale invariant theory Weyl weakened this condition, and required only that

$$\nabla_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu} \ , \tag{2.2.5}$$

where A_μ is a vector field. That is, in Weyl's theory only the angles, but not the lengths, are preserved under parallel transport.

If we assume that the torsion is zero, then Eq. (2.2.5) can be solved for the connection in terms of $g_{\mu\nu}$ and A_μ. The result is

$$\Gamma_{\mu\nu}{}^{\alpha} = \left\{ {\alpha \atop \mu\nu} \right\} - \frac{1}{2}(\delta^\alpha_\mu A_\nu + \delta^\alpha_\nu A_\mu - g_{\mu\nu} A^\alpha) \; , \tag{2.2.6}$$

where $\{ {\alpha \atop \mu\nu} \}$ is the usual Einstein connection. Notice that $\Gamma_{\mu\nu}{}^{\alpha}$ is gauge invariant and symmetric. From $\Gamma_{\mu\nu}{}^{\alpha}$ we can construct the conformally invariant curvature tensor:

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\nu\alpha}{}^{\gamma}\Gamma_{\mu\gamma}{}^{\beta} - \Gamma_{\mu\alpha}{}^{\gamma}\Gamma_{\nu\gamma}{}^{\beta} \; . \tag{2.2.7}$$

The conformally invariant Ricci tensor and the scalar curvature are given by contracting (2.2.7) with the metric. In $d+1$ dimensions the Ricci tensor is

$$\begin{aligned} R_{\mu\nu} \equiv R_{\beta\mu\nu}{}^{\beta} = R^E_{\mu\nu} + {} & \frac{1}{2}(d\ D_\mu A_\nu - D_\nu A_\mu + g_{\mu\nu} D_\alpha A^\alpha) \\ & + \frac{d-1}{4}(A_\mu A_\nu - g_{\nu\mu} A_\alpha A^\alpha) \; , \end{aligned} \tag{2.2.8}$$

and the scalar curvature is

$$R \equiv g^{\mu\nu} R_{\mu\nu} = R^E + d\ D_\alpha A^\alpha - \frac{d(d-1)}{4} A_\alpha A^\alpha \; , \tag{2.2.9}$$

where $R^E_{\mu\nu}$ and R^E are the usual Einstein Ricci tensor and scalar curvature, respectively, and D_μ is the covariant derivative with respect to $\{ {\alpha \atop \mu\nu} \}$. Notice that the Ricci tensor is not symmetric, since it contains the antisymmetric field strength tensor $F_{\mu\nu}$. From Eq. (2.2.8) its antisymmetric part is given by

$$R_{\mu\nu} - R_{\nu\mu} = \frac{d-1}{2} F_{\mu\nu} \; , \tag{2.2.10}$$

and $F_{\mu\nu}$ is defined as usual

$$F_{\nu\mu} = D_\nu A_\mu - D_\mu A_\nu = \partial_\nu A_\mu - \partial_\mu A_\nu \; , \tag{2.2.11}$$

because the connection is symmetric.

The covariant derivative with respect to the connection given by Eq. (2.2.6) acts on a scalar field φ and on a vector field V_μ of Weyl weight w in the following manner

$$\nabla_\mu \varphi = \partial_\mu \varphi + \frac{w}{2} A_\mu \varphi \ , \tag{2.2.12}$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha + \frac{w}{2} A_\mu V_\nu \ . \tag{2.2.13}$$

A field φ is of (Weyl) weight w if it transforms under conformal rescaling as

$$\varphi \rightarrow \Omega^{-w} \varphi \ . \tag{2.2.14}$$

Weyl invariance limits the form of the gravitational interaction. One cannot construct a purely gravitational action, that is, first order in the curvature, and reproduce Einstein's theory (except in the "trivial" 1+1 dimensional case, when it is a surface term). Mathematically this means that, under a conformal rescaling

$$\sqrt{|g|} \rightarrow \Omega^{d+1} \sqrt{|g|} \ , \tag{2.2.15}$$

where $g = \det\, g_{\mu\nu}$, and

$$R \rightarrow \Omega^{-2} R \ , \tag{2.2.16}$$

the Einstein action is not invariant:

$$I_E = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{|g|}\ R \rightarrow \frac{1}{16\pi G} \int d^{d+1}x \sqrt{|g|} R\ \Omega^{d-1} \ , \tag{2.2.17}$$

where G is Newton's constant. The action I_E is invariant only if $d = 1$, that is in 1+1 dimensions where it is the Euler characteristic. In 3+1 dimensions the Einstein action is not gauge invariant (an invariant gravitational action can be formed, but it has to be quadratic in R). On the other hand Maxwell's action

$$I_M = -\frac{1}{4} \int d^4x \sqrt{|g|} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \ , \tag{2.2.18}$$

is Weyl invariant. This is not surprising, because electromagnetism does not require any dimensional constant.

In order to reproduce Einstein's theory, many authors [8,19] have introduced a scalar field of unit weight, that is

$$\varphi \to \Omega^{-1}\varphi \ , \tag{2.2.19}$$

corresponding to its canonical dimension. Then the action

$$S = \int d^4x \sqrt{|g|} R \varphi^2 \ , \tag{2.2.20}$$

is indeed invariant. If φ acquires a nonzero value φ_0 in the vacuum, it yields the Einstein action with $\frac{1}{16\pi G} = \varphi_0^2$. Weyl invariance in 3+1 dimensions allows for a kinetic term for φ as well as a potential term, thus making it a dynamical field. However, it is not clear how to generate such a vacuum value in a theory that is not only without a scale but also not renormalizable.

In 2+1 dimensions one can also introduce a scalar field with Weyl weight 1 (not its canonical dimension) in order to construct a gauge invariant action, given by

$$S = \int d^3x \left(\sqrt{|g|} R\varphi + \frac{1}{2}\epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma + \frac{\lambda}{3}\sqrt{|g|}\varphi^3 \right) . \tag{2.2.21}$$

But in this case, Weyl invariance allows us to write a cubic potential term for φ, but not a kinetic term. Thus we do not expect φ to correspond to a dynamical degree of freedom. We further note that the action contains only a Chern-Simons term for the vector potential and no Maxwell term.

The equations of motion obtained from varying the action with respect to the fields and the metric are the following:

$$R + \lambda\varphi^2 = 0 \ , \tag{2.2.22}$$

$$\frac{1}{2\sqrt{|g|}}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta} = g^{\gamma\mu}\nabla_\mu\varphi \ , \tag{2.2.23}$$

$$(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\varphi + \{\mu \leftrightarrow \nu\} = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla_\alpha\nabla^\alpha)\varphi$$
$$+ \frac{\lambda}{6}\varphi^3 g_{\mu\nu} + \{\mu \leftrightarrow \nu\} \ . \tag{2.2.24}$$

It is interesting to note that, even though our Lagrangian does not contain derivatives of the scalar field, φ still obeys a Klein-Gordon equation, with the source as the Chern-Simons density

$$\nabla_\nu\nabla^\nu\varphi = \frac{1}{\sqrt{|g|}}\epsilon^{\alpha\beta\nu}F_{\alpha\beta}A_\nu \ , \tag{2.2.25}$$

using Eq. (2.2.23) and assuming the Bianchi identity on $F_{\mu\nu}$ (absence of a magnetic monopole). Actually φ is a derived field which obeys a first order differential equation of the form

$$V_\gamma g^{\gamma\mu}\nabla_\mu\varphi = 0 \ , \tag{2.2.26}$$

where V_μ is a vector field chosen such that

$$V_\gamma F_{\alpha\beta}\epsilon^{\alpha\beta\gamma} = 0 \ . \tag{2.2.27}$$

This means that φ is covariantly constant along a direction locally determined by the electric and magnetic fields. Before looking for solutions of these equations, we make the following simplifying observations. First, not all of the above three equations are independent. The trace of Eq. (2.2.24) is equivalent to Eqs. (2.2.22) and (2.2.23). This allows us to use only the last two equations when seeking solutions. Second, the Weyl vector terms can be incorporated into the energy-momentum tensor of the matter fields. As a result, Eq. (2.2.24) can be written in the usual form of the Einstein equation in Riemannian space:

$$G_{\mu\nu} \equiv R^E_{\mu\nu} - \frac{1}{2}R^E g_{\mu\nu} = T_{\mu\nu} \ . \tag{2.2.28}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \varphi^{-1}\Big((D_\mu\partial_\nu - g_{\mu\nu}D^\alpha\partial_\alpha)\varphi + A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - g_{\mu\nu}A^\alpha\partial_\alpha\varphi\Big) + (\frac{\lambda}{6}\varphi^2 - \frac{1}{4}A_\alpha A^\alpha)g_{\mu\nu} + \frac{1}{2}A_\mu A_\nu \ . \tag{2.2.29}$$

To proceed, we use the Weyl invariance to go into a gauge where

$$\varphi = \varphi_0 \ , \tag{2.2.30}$$

which is allowed as long as φ does not vanish anywhere. There, Eq. (2.2.23) is rewritten as

$$\frac{1}{\sqrt{|g|}}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta} = \varphi_0 A^\gamma \ , \tag{2.2.31}$$

from which it follows that $D_\alpha A^\alpha = 0$. From Eq. (2.2.31) one also finds that A_ν satisfies

$$D^\alpha D_\alpha A_\nu - \frac{\varphi_0^2}{4}A_\nu + A^\alpha R^E_{\alpha\nu} = 0 \ , \tag{2.2.32}$$

the field equation of a massive vector field. A similar result has been found for the Abelian Chern-Simons theory on Minkowski spacetime in refs. 20 and 21. In this gauge the energy-momentum tensor of Eq. (2.2.29) simplifies to

$$T_{\mu\nu} = (\frac{\lambda}{6}\varphi_0^2 - \frac{1}{4}A_\alpha A^\alpha)g_{\mu\nu} + \frac{1}{2}A_\mu A_\nu \ , \tag{2.2.33}$$

a form reminiscent of a fluid. It is not quite perfect since the pressure p and the density ρ depend on $A_\alpha A^\alpha$, as well as on φ_0^2. Furthermore the fluid has a velocity vector proportional to A_μ which obeys the constraint (2.2.31). We will study the properties of this fluid in greater detail in the context of exact solutions.

If the electromagnetic field is zero, *i.e.* $E = B = 0$, we find from Eq. (2.2.23)

$$\varphi = \varphi_0 e^{-\lambda/2} \ , \tag{2.2.34}$$

with

$$A_\mu = \partial_\mu \chi \ , \tag{2.2.35}$$

and the Einstein equation reduces to

$$G_{\mu\nu} = -\frac{1}{4}(A_\mu A_\nu + D_\mu A_\nu + D_\nu A_\mu) + g_{\mu\nu}(\frac{1}{2}D_\alpha A^\alpha + \frac{\lambda}{6}\varphi^2) \ . \tag{2.2.36}$$

Clearly, when the Weyl invariance is gauge fixed to a constant φ, we obtain

$$G_{\mu\nu} = g_{\mu\nu}\frac{\lambda}{6}\varphi^2 \ , \tag{2.2.37}$$

reducing the space to an Einstein space with cosmological constant. In the following, we discuss stationary solutions.

2.3 Stationary Solutions

In the stationary case the most general form of the line element in two component notation is [22]

$$ds^2 = -N^2(dt + K_i dx^i)^2 + \gamma_{ij}dx^i dx^j , \tag{2.3.1}$$

where N, K_i and γ_{ij} depend only on the spatial coordinates, x^i, and $i, j = 1, 2$. That is, the metric components are

$$g_{00} = -N^2 \ , \quad g_{0i} = -N^2 K_i \ , \quad g_{ij} = \gamma_{ij} - N^2 K_i K_j \ , \tag{2.3.2}$$

with inverse components

$$g^{00} = -\frac{1}{N^2} + \gamma^{ij}K_i K_j \ , \quad g^{0i} = -\gamma^{ij}K_j = -K^i \ , \quad g^{ij} = \gamma^{ij} \ , \tag{2.3.3}$$

$$\sqrt{|g|} = N\sqrt{\gamma} \ , \quad \gamma = \det \gamma_{ij} \ , \quad N > 0 \ . \tag{2.3.4}$$

The remaining reparametrization gauge freedom, $t \to t + \Lambda(r)$, can be fixed by setting $D_i K^i = 0$, where D_i is the two dimensional covariant derivative with respect to γ_{ij}. In this notation the components of the Einstein tensor are

$$G_{00} = \frac{N^2}{2}\ {}^2R + \frac{3}{8}N^4 K^{ij}K_{ij} \ , \tag{2.3.5}$$

$$G_0{}^i = -\frac{1}{2N} D_j(N^3 K^{ij}) \; , \tag{2.3.6}$$

$$G^{ij} = \frac{1}{2} N^2 K^{ik} K^j{}_k - \frac{1}{N}(D^i \partial^j - \gamma^{ij} D^2) N - \frac{1}{8} N^2 \gamma^{ij} K^{kl} K_{kl} \; , \tag{2.3.7}$$

where $K_{ij} = D_i K_j - D_j K_i = \partial_i K_j - \partial_j K_i$, and ${}^2R$ is the two dimensional curvature.

We choose spatial conformal coordinates. This can be done locally without loss of generality in a two dimensional space:

$$\begin{aligned} \gamma_{ij} &= \Phi \delta_{ij} \; , \quad (\Phi > 0), \\ dl^2 &= \Phi(dx^2 + dy^2) \; . \end{aligned} \tag{2.3.8}$$

In these coordinates the $D^i K_i = 0$ condition reduces to $\partial_i K_i = 0$. This allows us to write $K_i = \epsilon_{ij} \partial_j V$, where ϵ_{ij} is the two dimensional Levi-Cività tensor with $\epsilon_{12} = -\epsilon_{21} = 1$.

With this form of the metric, the field equations (2.2.22), (2.2.23) and (2.2.24) in the $\varphi = \varphi_0$ gauge become

$$\epsilon_{ij} \partial_i A_j = -\frac{N\varphi_0}{2} \Big(\frac{\Phi A_0}{N^2} + K_j (A_j - A_0 K_j) \Big) \; , \tag{2.3.9}$$

$$\epsilon_{ij} \partial_j A_0 = \frac{N\varphi_0}{2} (A_i - A_0 K_i) \; , \tag{2.3.10}$$

$$\frac{1}{\Phi} \Delta \ln \Phi - \frac{3N^2}{2\Phi^2} (\Delta V)^2 = (\frac{\lambda}{3} \varphi_0^2 - \frac{1}{2} A_\alpha A^\alpha) - \frac{1}{2N^2} A_0^2 \; , \tag{2.3.11}$$

$$\frac{1}{N} \partial_j (\frac{N^3}{\Phi} \epsilon_{ij} \Delta V) = A_0 (A_i - A_0 K_i) \; , \tag{2.3.12}$$

$$\begin{aligned} &\frac{1}{N\Phi^2} (D_i \partial_j N - \delta_{ij} \Delta N) - \frac{N^2}{4\Phi^3} \delta_{ij} (\Delta V)^2 \\ &= -(\frac{\lambda}{6} \varphi_0^2 - \frac{1}{4} A_\alpha A^\alpha) g_{ij} - \frac{1}{2} A_i A_j \; , \end{aligned} \tag{2.3.13}$$

where Δ is the two dimensional flat Laplacian.

The solutions can be characterized by the nonvanishing components of $\mathrm{F}_{\mu\nu}$. In analogy with electrodynamics we call the space-space component the

magnetic field, $B = -\frac{1}{2\sqrt{\gamma}}\epsilon_{ij}F_{ij}$, and the space components of the electric field, $E_i = F_{0i}$. We have the following cases.

a) $B = 0,\ E_i = 0$. We have already seen that this case reduces to Einstein gravity in flat or in de Sitter space [2].

b) $B \neq 0,\ E_i = 0$. This means that A_0 is constant and from Eqs. (2.3.10) we have $A_i = A_0 K_i$. Further, since B is different from zero, A_0 itself cannot vanish. Thus $A_i = A_0 \epsilon_{ij}\partial_j V$ allows us to rewrite Eq. (2.3.9) as

$$\Delta V = \frac{\Phi\ \varphi_0}{2N}\ , \tag{2.3.14}$$

which by comparing with Eq. (2.3.12) leads to

$$N = N_0 = \text{constant}. \tag{2.3.15}$$

Note that our gauge condition (2.2.30) still allows us to make constant gauge transformations to rescale N_0 (and φ_0). We fix it by choosing $N_0 = 1$. In this purely magnetic case we were able to find all the solutions of the field equations.

c) $E_i \neq 0,\ B = \text{anything}$. In this case, when the electric field is not zero, we did not find any static solutions. We can say however that if only one component of the electric field is nonzero, say E_1, then the solutions depend on only one spatial coordinate x_1.

In the next two sections we solve the field equations in the purely magnetic case with $B \neq 0$ (Sect. 2.4) and with $B = 0$ (Sect. 2.5). In the latter case, when our theory reduces to the Einstein case, we find the general solution of the equations of motion. In Sect. 2.6 we continue with studying our solutions, obtained for the pure magnetic case. We show that in the axisymmetric case they correspond to 2+1 dimensional versions of the known 3+1 dimensional spacetimes of Gödel and Taub-NUT.

2.4 Solutions with Magnetic Field

In this section we solve Eqs. (2.3.9)–(2.3.13) in the purely magnetic case ($B \neq 0,\ E_i = 0$). At the end of the previous section we showed that in this case $A_0 =$ constant, $A_i = A_0 K_i$ and $N_0 =$ constant. Using these results we find that Eq. (2.3.10) is trivially solved and the remaining equations are

$$\epsilon_{ij}\partial_i A_j = -\frac{1}{2}\varphi_0 A_0 \Phi \ , \tag{2.4.1}$$

$$-\frac{1}{\Phi}\Delta \ln \Phi + \frac{3}{2}\left(\frac{\Delta V}{\Phi}\right)^2 = -\frac{\lambda}{3}\varphi_0^2 + \frac{1}{2}A_0^2 \ , \tag{2.4.2}$$

$$\partial_j\left(\frac{\Delta V}{\Phi}\right) = 0 \ , \tag{2.4.3}$$

$$\left(\frac{\Delta V}{\Phi}\right)^2 = \frac{2\lambda}{3}\varphi_0^2 + A_0^2 \ , \tag{2.4.4}$$

where we have set $N_0 = 1$. From the above equations one finds that Φ satisfies a Liouville equation

$$\Delta \ln \Phi = \beta \Phi \ , \tag{2.4.5}$$

where $\beta = \frac{1}{4} + \frac{2}{3}\lambda$. All the other quantities can be expressed in terms of the solutions of this equation and the constants φ_0 and λ as follows

$$A_0^2 = (\frac{1}{4} - \frac{2}{3}\lambda)\varphi_0^2 \ , \tag{2.4.6}$$

$$\Delta V = \frac{1}{2}{\varphi_0}^2\Phi \ , \tag{2.4.7}$$

$$A_i = A_0 \epsilon_{ij}\partial_j V \ . \tag{2.4.8}$$

Note that because A_0^2 is nonnegative only the $\lambda \leq \frac{3}{8}$ values are allowed. This leads to $\beta \leq \frac{1}{2}$, that is, it can be either negative or positive.

From the above equations one finds that the magnetic field,

$$B = \frac{A_0}{\Phi}\Delta V = \frac{A_0}{2}\varphi_0^2 \ , \tag{2.4.9}$$

the length of the Weyl gauge field,

$$A_\mu A^\mu = -A_0^2 \ , \tag{2.4.10}$$

and the Einstein scalar curvature,

$$\begin{aligned} R &= {}^2R + \frac{1}{4}K^{ij}K_{ij} \\ &= -\frac{1}{\Phi}\Delta \ln \Phi + \frac{1}{2\Phi^2}(\Delta V)^2 = -\frac{\varphi_0^2}{8}(1 + \frac{16\lambda}{3}) \ , \end{aligned} \tag{2.4.11}$$

are constant. We note that in this case, the perfect fluid analogy mentioned in Sect. 2.2 is complete, because as follows from Eqs. (2.2.30) and (2.4.10) the energy-momentum tensor is just

$$T_{\mu\nu} = \frac{\varphi_0^2}{16} g_{\mu\nu} + \frac{1}{2} A_\mu A_\nu \ . \tag{2.4.12 }$$

The normalized velocity is then

$$u_\mu = \frac{A_\mu}{|A_0|} \ , \tag{2.4.13}$$

that is,

$$u_0 = 1 \text{ and } u_i = \epsilon_{ij}\partial_j V \ . \tag{2.4.14}$$

The equation of state relating the density ρ to the pressure p is given by

$$\rho = \begin{cases} \frac{8\beta-3}{8\beta} p & \text{if } \beta \neq 0 \\ \frac{\varphi^2}{16} \ , \ p = 0 & \text{otherwise,} \end{cases} \tag{2.4.15}$$

where the $p = 0$ case corresponds to dust. We note that the (weak and dominant) energy conditions [23], stemming from demanding causality, lead to the condition $\rho \geq |p|$, which gives further restriction on β: either $\beta < 0$ and $p \geq 0$ or $0 < \beta \leq \frac{3}{16}$. In addition to this equation of state, the velocity vector u_μ obeys the further equation

$$u^\gamma = \frac{1}{\varphi_0\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma} \partial_\alpha u_\beta \ , \tag{2.4.16}$$

which is indicative of rotation. In particular this means that the velocity obeys a massive Klein-Gordon equation, and that the fluid is incompressible.

Depending on how we choose the value of λ, R can be positive or negative. Positive R corresponds to a compact space-time manifold (*i.e.*, "closed universe"). In this case the solutions can be characterized by topological invariants of the manifold. For negative R, that is, for noncompact manifolds (or "open universe"), one can define the energy and the angular momentum of the solution.

We also find that our solution is conformally flat. In three dimensional space the Weyl tensor is always zero in the absence of matter. However there is another tensor, the Cotton tensor [24],

$$C^{\mu\nu} = \frac{1}{2\sqrt{|g|}}(\epsilon^{\alpha\beta\mu}\nabla_\beta R^\nu_\alpha + \epsilon^{\alpha\beta\nu}\nabla_\beta R^\mu_\alpha) , \tag{2.4.17}$$

which plays the same role as the Weyl tensor in higher dimensional spaces. It is symmetric, covariantly conserved and vanishes if and only if the spacetime is conformally flat. For example all the vacuum solutions as well as the point particle and rotating solutions of Deser *et al.* [2] are conformally flat. In our case $C^{\mu\nu}$ is vanishing; that is, our solution is also conformally flat.

The general solution of the Liouville equation is given in terms of two complex functions $f(z)$ and $g(\bar{z})$,

$$\Phi = -\frac{2}{\beta}\frac{\partial_z f \partial_{\bar{z}} g(\bar{z})}{(f(z) - g(\bar{z}))^2} , \tag{2.4.18}$$

where $f(z)$ and $g(\bar{z})$ are such that they give real positive values to Φ. Explicit forms of f and g, that satisfy this requirement, are known [25]. As the simplest case, we will consider axial symmetric solutions. But first we discuss the $B = E_i = 0$ case.

2.5 Solutions with no Electromagnetic Field: Einstein Gravity

Although we have already shown that this case reduces to Einstein gravity with a cosmological term, it is instructive to elaborate on the form of the static solutions. The equations become

$$\frac{1}{\Phi}\Delta \ln \Phi - \frac{3N^2}{2}\Big(\frac{\Delta V}{\Phi}\Big)^2 = \frac{\lambda}{3}\varphi_0^2 \ , \tag{2.5.1}$$

$$\partial_j\Big(N^3\frac{\Delta V}{\Phi}\Big) = 0 \ , \tag{2.5.2}$$

$$\frac{1}{N\Phi}(D_i\partial_j N - \delta_{ij}\Delta N) - \frac{N^2}{4}\Big(\frac{\Delta V}{\Phi}\Big)^2\delta_{ij} = -\frac{\lambda}{6}\varphi_0^2\delta_{ij} \ . \tag{2.5.3}$$

To solve these equations, we have to consider the $N =$ constant and $N \neq$ constant cases separately.

a) $N =$ constant:

Without loss of generality we can set $N = 1$, then from Eqs. (2.5.1) and (2.5.2) we obtain

$$\frac{1}{\Phi}\Delta \ln \Phi = \frac{4\lambda}{3}\varphi_0^2 = \beta \ , \tag{2.5.4}$$

$$\partial_j\Big(\frac{\Delta V}{\Phi}\Big) = 0 \ , \tag{2.5.5}$$

$$\Big(\frac{\Delta V}{\Phi}\Big)^2 = \frac{2\lambda}{3}\varphi_0^2 \ . \tag{2.5.6}$$

Notice that these equations are similar to the ones we obtained for the pure magnetic case. The field Φ satisfies a Liouville equation (Eq. (2.5.4)), and, from Eqs. (2.5.5) and (2.5.6), V can be obtained in terms of the solutions of that equation.

However there are differences between the two cases. The above equations have solutions only for nonnegative λ as follows from Eq. (2.5.6) (in the magnetic case we have $\lambda \leq \frac{3}{8}$). This implies (Eq. (2.5.4)) that β has to be nonnegative as well (in the magnetic case $\beta \leq \frac{1}{2}$). As we shall see in the next section, the Liouville equation has qualitatively different classes of solutions

depending on the sign of β. Here we note that because the $\beta = 0$ case corresponds to flat space-time solutions, *i.e.*, $\Delta \ln \Phi = 0$ and $\Delta V = 0$, we do not consider it here. Instead, we discuss the more interesting $\beta > 0$ and $\beta < 0$ cases. We can say that because $\beta > 0$ in this case, we obtain only one class of solutions while in the magnetic case we have both classes.

b) $N \neq$ constant:

To solve our equations in the $N \neq$ constant case we follow similar techniques used in ref. 2 to obtain multiparticle solutions for the Einstein equations with nonzero cosmological constant. Note that our solutions are more general, since the solutions of ref. 2 correspond to nonrotating sources ($\alpha = N^3 \frac{\Delta V}{\Phi} = 0$ case in our notation), while ours describe rotating sources as well.

We start with separating Eq. (2.5.3) into the spatial trace

$$\frac{1}{N\Phi}\Delta N + \frac{1}{2N^4}\alpha^2 = \frac{\lambda}{3}\varphi_0^2 \, , \tag{2.5.7}$$

and into the traceless part

$$\partial_i M_j + \partial_j M_i - \delta_{ij}\partial_k M_k = 0 \, , \tag{2.5.8}$$

where $M_i = \frac{1}{\Phi}\partial_i N$, and

$$\alpha = N^3 \frac{\Delta V}{\Phi} = \text{constant}, \tag{2.5.9}$$

as follows from Eq. (2.5.2). Note that if we define a complex function $M = M_1 + iM_2 = \frac{1}{\Phi}\partial_{\bar{z}} N$, then Eq. (2.5.8) becomes the Cauchy Riemann equation for M, which is solved by $M = M(z)$. Equations (2.5.1) and (2.5.7) become

$$\frac{1}{\Phi}\Delta\Phi - \frac{3}{2N^4}\alpha^2 = \frac{\lambda}{3}\varphi_0^2 \, , \tag{2.5.10}$$

and

$$\frac{1}{\Phi}\partial_{\bar{z}}\partial_z N + \frac{1}{2N^3}\alpha^2 = \frac{\lambda}{3}N\varphi_0^2 \, . \tag{2.5.11}$$

After multiplication by $\partial_{\bar z} N$ and integration with respect to $\bar z$, Eq. (2.5.11) becomes

$$M(z)\partial_z N - \Big(\frac{\lambda\varphi_0^2}{6}N^2 + \frac{\alpha^2}{4}N^{-2}\Big) = \frac{1}{2}\epsilon(z) \ , \tag{2.5.12}$$

where $\epsilon(z)$ is an arbitrary integration "constant". If we introduce a real parameter

$$\zeta = \frac{1}{2}\Big(\int^{z} \frac{dw}{M(w)} + \int^{\bar z} \frac{d\bar w}{M(\bar w)}\Big) \ , \tag{2.5.13}$$

then Eq. (2.5.12) becomes

$$\partial_\zeta N = \Big(\frac{\lambda\varphi_0^2}{3}N^2 + \frac{\alpha^2}{2}N^{-2}\Big) + \epsilon(z) \ . \tag{2.5.14}$$

This is a first order ordinary differential equation for $N(\zeta)$; the solutions are real only if ϵ is a real constant, and they are given by standard integrals through

$$\int_{N(\zeta_0)}^{N(\zeta)} \frac{dN}{\frac{\lambda\varphi_0^2}{3}N^2 + \frac{\alpha^2}{2}N^{-2} + \epsilon} = \zeta - \zeta_0 \ . \tag{2.5.15}$$

The solution for the spatial conformal factor Φ is given in terms of the solutions of Eq. (2.5.15), $M(z)$ and the constant parameters λ, φ_0, α and ϵ:

$$\Phi = \frac{1}{2M(z)\bar M(\bar z)}\Big(\frac{\lambda\varphi^2}{3}N^2 + \frac{\alpha^2}{2}N^{-2} + \epsilon\Big) \ , \tag{2.5.16}$$

as one can see from Eq. (2.5.1) and the definition of $M(z)$.

Thus in the $N \neq$ constant case, the solution is given by Eqs. (2.5.9), (2.5.15) and (2.5.16) in terms of an arbitrary holomorphic function, $M(z)$. Once $M(z)$ is specified, the explicit forms of $N(x)$, $V(x)$ and $\Phi(x)$ are obtained by the above equations.

2.6 Stationary Axial Symmetric Solutions

In the previous sections we showed that stationary solutions of 2+1 dimensional Weyl theory can be found. In the cases we have studied, the metric is

given by Eq. (2.3.1) with $N = 1$ and $\gamma_{ij} = \Phi\delta_{ij}$ and the problem reduces to the solution of a Liouville equation (Eqs. (2.4.5) and (2.5.4)) for the spatial conformal factor Φ. This means that the spatial part of the spacetime is of constant curvature (negative if $\beta > 0$ and positive if $\beta < 0$). In order to simplify our discussions, we consider only axial symmetric solutions. The most general such solutions are given in terms of two real parameters, a and ν , and the radial coordinate $r = \sqrt{(x^2 + y^2)}$ as follows [25],

$$\Phi = \frac{8\nu^2 a^{2\nu}}{|\beta|}\frac{r^{2\nu-2}}{(r^{2\nu} + a^{2\nu})^2} \quad \text{if } \beta < 0\ , \tag{2.6.1}$$

and

$$\Phi = \frac{8\nu^2 a^{2\nu}}{|\beta|}\frac{r^{2\nu-2}}{(r^{2\nu} - a^{2\nu})^2} \quad \text{if } \beta > 0\ , \tag{2.6.2}$$

and the $\beta = 0$ case would correspond to the flat solution, $\Delta\Phi = \Delta V = 0$. The parameter a can be absorbed into r by introducing $\xi = \frac{r}{a}$. The other parameter, ν, has to be nonzero; otherwise Φ would be zero. Because of the invariance of Φ under inversion, $r \to \frac{1}{r}$, it is enough to consider the $\nu > 0$ case.

In the magnetic case, space-time components $K_i = \epsilon_{ij}\partial_j V$ of the metric are given by Eqs. (2.4.4) and (2.4.6):

$$K_i = \epsilon_{ij}\frac{x^j}{r^2}\frac{\varphi_0}{\beta}\left(\nu - 1 + \frac{b}{2} - \frac{2\nu r^{2\nu}}{r^{2\nu} \pm a^{2\nu}}\right)\ . \tag{2.6.3}$$

In the $E_i = B = 0$ case, as one can see from Eq. (2.5.6), K_i is given by the same expression, but with different numerical factors. We choose the integration constant b in Eq. (2.6.3) such that K_i is nonsingular at the origin; that is, we set $b = 2(\nu - 1)$. Then, the line element in spherical coordinates reads

$$ds^2 = -(dt + \frac{2\nu}{\beta}\varphi_0\frac{r^{2\nu}}{(r^{2\nu} \pm a^{2\nu})}d\psi)^2 + \frac{8n^2}{|\beta|}\frac{r^{2\nu}}{(r^{2\nu} \pm a^{2\nu})^2}(dx^2 + x^2 d\psi^2)\ . \tag{2.6.4}$$

Note that the metric is singular at $r \to \infty$, because Φ vanishes there. In the $\beta > 0$ case it is also singular at $r = a$. The latter is very much like the case of solutions of Einstein's equations that describe a rotating fluid. Notice also that because the diffeomorphism invariant quantities such as the scalar curvature and the length of the Weyl vector are nonsingular everywhere (they are constants) these are only coordinate singularities.

Since we have an explicit solution, we can complete our perfect fluid analogy discussed previously by calculating the normalized velocity. From Eq. (2.4.13) we find that in spherical coordinates

$$u_0 = 1 \ , \quad u_r = 0 \text{ and } \ u_\psi = \frac{2\nu}{\beta}\varphi_0 \frac{r^{2\nu}}{r^{2\nu} \pm a^{2\nu}} \ , \tag{2.6.5}$$

where we have used that $A_0 K_i = A_i$ in the purely magnetic case. Thus our solution corresponds to circular flow with vorticity, $v^\alpha = -\frac{1}{2\sqrt{|g|}}\epsilon^{\alpha\beta\gamma}\partial_\beta v_\gamma$:

$$v_0 = \frac{\varphi_0^2}{2} \ , \quad v_i = 0 \ ,$$

that is, only the time component is nonzero, and it is constant.

Before we proceed with the discussion of the space-time structure of our solution, we note that the $\nu \neq 1$ case always can be brought into the form of the $\nu = 1$ case by rescaling the radial coordinate. As we shall see, the only difference is that the range of the angular coordinate will change. To see this let us define a new radial coordinate,

$$\xi = \left(\frac{r}{a}\right)^\nu .$$

In terms of ξ the line element becomes

$$ds^2 = -(dt + \frac{2\varphi_0}{\beta}\frac{\xi^2}{(\xi^2 \pm 1)}d\psi')^2 + \frac{8}{|\beta|}\frac{1}{(\xi^2 \pm 1)^2}(d\xi^2 + \xi^2 d\psi'^2) \ , \tag{2.6.6}$$

where the new angular coordinate $\psi' = \nu\psi$ ranges from 0 to $2\pi\nu$ (if $0 \leq \psi \leq 2\pi$); that is, the points with ψ' and $\psi' + 2\pi\nu$ are identified.

In the following we consider the positive and negative β cases separately. We consider the $\nu = 1$ case to simplify our discussion.

a) $\beta > 0$:

The solution of the Liouville equation is given by Eq. (2.6.2). The metric in this case is given by Eq. (2.6.5) with the lower sign, and it is singular at $\xi = 1$. Hence, we have to consider the $\xi < 1$ and $\xi > 1$ cases separately. If $\xi < 1$, the change of the radial coordinate $\tilde{\xi} = 2\xi/(1-\xi^2)$ gives the following line element

$$ds^2 = -(dt - \frac{\varphi_0}{\beta}(\sqrt{(\tilde{\xi}^2+1)} - 1)d\psi)^2 + \frac{2}{\beta}(\frac{d\tilde{\xi}^2}{1+\tilde{\xi}^2} + \tilde{\xi}^2 d\psi^2) , \qquad (2.6.7)$$

where $-\infty < t < \infty$, $0 \leq \psi \leq 2\pi$ and $0 \leq \tilde{\xi} < \infty$. A final, "hyperbolic," formula is obtained by defining $\tilde{\xi} = \sinh\sigma$,

$$ds^2 = -(dt - \frac{\varphi_0}{\beta}(\cosh\sigma - 1)d\psi)^2 + \frac{2}{\beta}(d\sigma^2 + \sinh^2\sigma d\psi^2) , \qquad (2.6.8)$$

and $0 \leq \sigma < \infty$.

Similarly for $\xi > 1$, the change of the radial coordinate $\tilde{\xi} = \xi/(\xi^2-1)$ gives

$$ds^2 = -(dt + \frac{\varphi_0}{\beta}(\sqrt{\tilde{\xi}^2+1} + 1)d\psi)^2 + \frac{2}{\beta}(\frac{d\tilde{\xi}^2}{\tilde{\xi}^2+1} + \tilde{\xi}^2 d\psi^2) , \qquad (2.6.9)$$

where $-\infty < t < \infty$, $0 \leq \tilde{\xi} < \infty$ and $0 \leq \psi \leq 2\pi\nu$. The final " hyperbolic " formula is obtained by defining $\tilde{\xi} = \sinh\sigma$:

$$ds^2 = -(dt + \frac{\varphi_0}{\beta}(\cosh\sigma + 1)d\psi)^2 + \frac{2}{\beta}(d\sigma^2 + \sinh^2\sigma d\psi^2) , \qquad (2.6.10)$$

where $0 \leq \sigma < \infty$. We note that this is the metric of the Gödel universe [26]. A similar solution in 2+1 dimensions has been found for topological gravity [27].

We have already mentioned that our solution is conformally flat (the Cotton tensor is vanishing). Here we show that one can find a set of coordinates, in terms of which, the form of the metric reduces to the flat solution, with periodic time. In the $\nu = 1$ case, the metric is given by

$$ds^2 = -(dt + K_i dx^i)^2 + \Phi dx^i dx^i \ , \tag{2.6.11}$$

where K_i and Φ are given by Eqs. (2.6.3) and (2.6.1)–(2.6.2), respectively. Let us introduce new coordinates,

$$\rho^2 = |r^2 - a^2| \ , \tag{2.6.12}$$

and denote the corresponding angular coordinate χ. The line element in terms of these coordinates is

$$ds^2 = -(dt \pm \frac{2\varphi_0}{\beta} d\chi)^2 + \frac{8a^2}{|\beta|} \frac{1}{\rho^4}(d\rho^2 + \rho^2 d\chi^2) \ , \tag{2.6.13}$$

where one has the lower sign if $r < a$, and the upper sign if $r > a$. Let us make one more coordinate transformation:

$$\rho' = \frac{1}{\rho} \ . \tag{2.6.14}$$

Then the metric reduces to the familiar form

$$\begin{aligned} ds^2 &= -(dt \pm \frac{2\varphi_0}{\beta} d\chi) + \frac{8a^2}{|\beta|}(d\rho'^2 + \rho'^2 d\chi^2) \\ &= -dt'^2 + \frac{8a^2}{|\beta|}(d\rho'^2 + \rho'^2 d\chi^2) \ , \end{aligned} \tag{2.6.15}$$

where we have introduced a new periodic time coordinate

$$t' = \begin{cases} t - \frac{2\varphi_0}{\beta}\chi , & \text{if } r < a \\ t + \frac{2\varphi_0}{\beta}\chi & \text{if } r > a \ . \end{cases} \tag{2.6.16}$$

The metric given by Eq. (2.6.15) is flat, but the range of the coordinates is unusual. First of all, t and $t \pm \text{integer} \times \frac{4\pi\varphi_0}{\beta}$ have to be identified. Second,

$0 \le \rho' < \infty$, but the coordinates cover only the $0 \le r < a$ or the $a < r < \infty$ part of the spacetime.

b) $\beta < 0$:

Let us make the following coordinate transformation:

$$0 \le \sin^2\frac{\theta}{2} = \frac{\xi^2}{\xi^2+1} \le 1 \ , \ T = \frac{t}{\varphi_0} \ , \tag{2.6.17}$$

$$0 \le \theta \le \pi \ , \quad 0 \le \xi \le \infty \ , \quad -\infty \le T \le \infty \ . \tag{2.6.18}$$

In these coordinates the line element has the form

$$ds^2 = -\varphi_0^2(dT + \frac{1}{\beta}\sin^2\frac{\theta}{2}d\psi)^2 + \frac{2}{|\beta|}(d\theta^2 + \sin\theta^2 d\psi^2) \ . \tag{2.6.19}$$

As we have already mentioned the spatial part of the metric is a two dimensional sphere. And because the curvature of this sphere is ${}^2R = -\Delta\ln\Phi/\Phi = |\beta|$, the factor in front of the spatial part, $2/|\beta|$, is the square of the radius. The Euler characteristic, $\frac{1}{4\pi}\int d^2x\sqrt{\gamma}\ {}^2R = 2$, is that of the sphere.

The metric is regular everywhere, except at $\theta = \pi$, where it has a Dirac string type singularity. One can remove this singularity by introducing a new time coordinate

$$T' = T + \frac{1}{|\beta|}\psi \ . \tag{2.6.20}$$

The metric then becomes

$$ds^2 = -\varphi_0^2(dT' - \frac{1}{|\beta|}\cos^2\frac{\theta}{2}\ d\psi)^2 + \frac{2}{|\beta|}(d\theta^2 + \sin^2\theta d\psi^2) \ . \tag{2.6.21}$$

This is regular at $\theta = \pi$, but not at $\theta = 0$. One can therefore use the coordinates $(T,\ \theta,\ \psi)$ to cover the northern hemisphere $(0 \le \theta \le \frac{\pi}{2})$, and the coordinates $(T',\ \theta,\ \psi)$ at the southern hemisphere $(\frac{\pi}{2} \le \theta \le \pi)$. Because ψ is an angular variable with period 2π, T and T' have to be identified with the same period. This leads to a quantization condition between the energy of a field, defined

on the sphere and the curvature (or radius) of the sphere. Namely, in order for the field to be regular, single valued with time dependence $e^{i\omega T}$, the equality,

$$\frac{\omega}{|\beta|} = \frac{\omega}{2R} = \text{integer}, \tag{2.6.22}$$

has to hold.

It is interesting to notice that the form of our metric is the 3+1 dimensional Taub-NUT solution for a fixed radial coordinate [28]. Because we can choose $r \to \infty$, we can think of our solution as the boundary of the Taub-NUT solution. The topology of the boundary (and of any r = constant surface) is locally S^3, but globally it is that of a deformed sphere in the following sense. The Killing vector field defines $\frac{\partial}{\partial T}$ a nontrivial Hopf fibration: $S^3 \to S^2$, where the S^2 is parametrized by θ and ψ, and the fibres are circles. Thus the topolgy is a "twisted product" $S^1 \times S^2$. Thus the solution can be characterized with the Hopf invariant of the mapping from the compact three dimensional space-time manifold to the two dimensional spatial part, and with the Euler characteristic of the latter.

As in the positive β case the solution is not only conformally flat, but also can be brought into flat form, with unconventional range of the coordinates. In the $\nu = 1$ case the metric is given by

$$ds^2 = -(dt + K_i dx^i)^2 + \Phi dx^i dx^i \ , \tag{2.6.23}$$

where K_i and Φ are given by Eqs. (2.6.3) and (2.6.1)-(2.6.2), respectively. Let us introduce new coordinates,

$$\rho^2 = r^2 + a^2 \ , \tag{2.6.24}$$

and denote the corresponding angular coordinate χ. The line element in terms of these coordinates is

$$ds^2 = -(dt + \frac{2\varphi_0}{\beta} d\chi)^2 + \frac{8a^2}{|\beta|} \frac{1}{\rho^4}(d\rho^2 + \rho^2 d\chi^2) . \tag{2.6.25}$$

Let us make one more coordinate transformation

$$\rho' = \frac{1}{\rho} . \tag{2.6.26}$$

The metric then reduces to the familiar form

$$\begin{aligned} ds^2 &= -(dt + \frac{2\varphi_0}{\beta} d\chi) + \frac{8a^2}{|\beta|}(d\rho'^2 + \rho'^2 d\chi^2) \\ &= -dt'^2 + \frac{8a^2}{|\beta|}(d\rho'^2 + \rho'^2 d\chi^2) , \end{aligned} \tag{2.6.27}$$

where we have introduced a new periodic time coordinate,

$$t' = t + \frac{2\varphi_0}{\beta}\chi . \tag{2.6.28}$$

The metric given by Eq. (2.6.20) is flat, but again the range of the coordinates is unusual, t and $t \pm$ integer $\times$ $\frac{4\pi\varphi_0}{\beta}$ are identified, and $0 < \rho' < 1/a$.

Notice that in both cases, the metric can be transformed into the flat Minkowski form, if we introduce a new periodic time coordinate. Because of this feature, we suspect that our theory is equivalent to a finite temperature one.

We have discussed our stationary solutions in the axial symmetric case. We found that in the positive and negative β cases the solutions have different properties. In the Einstein case ($E_i = B = 0$) one has only the solutions that correspond to 2+1 dimensional Gödel universes, because $\beta > 0$. In the case of nonzero magnetic field ($B \neq 0$, $E_i = 0$) however, one can have the solutions that describe 2+1 dimensional Taub-NUT spacetimes as well, because β can be both negative and positive ($\beta \leq \frac{1}{2}$).

We have also observed in Sect. 2.4, that in the latter case our solution is similar to that for a rotating "Chern-Simons" fluid. As we have shown, this solution is causal only if $\beta \leq \frac{3}{16}$. This means that the fluid analogy holds for the Taub-NUT case, and for the Gödel case with $\beta \leq \frac{3}{16}$, but in the latter the pressure is negative.

CHAPTER 3
RENORMALIZABILITY OF $D = 3$ TMG

3.1 Introduction

By now it is well-known that perturbative quantum gravity in four spacetime dimensions suffers from the problem of nonrenormalizability. This may be cured by going to lower dimensions, but in this case the theory is much less interesting, because gravity in $D < 4$ in the absence of matter has no dynamical degrees of freedom. Recently Deser and Yang [9] have shown that topologically massive gravity [21] in three dimensions has the possibility of being renormalizable. Because this theory is massive, it does possess dynamics even in three dimensions. Although such a three dimensional theory clearly does not describe the universe in which we live, it would be of great theoretical interest to find such a renormalizable theory with the symmetries of gravity.

Deser and Yang have shown, by using an unusual parametrization of the metric, that TMG has power counting behavior consistent with renormalizability. This by itself does not establish the result, because one needs to show that the theory may be regulated in such a way to preserve both the theory's gauge invariance and the desirable power counting behavior. We will apply the newly discovered nonlocal regularization [11] to this theory. We will show that using this regulator, the possible obstacle to renormalizability discussed by Deser and Yang does not appear, and that if this technique is valid, the theory is in fact renormalizable to all orders. However, the validity of this

technique depends on the existence of a functional integration measure which is invariant under the nonlocally generalized gauge symmetry, which has not at this time been proven.

3.2 Power-Counting Renormalizability and Gravity

Most gravity theories are not power-counting renormalizable due in part to the presence of a coupling with negative mass dimension. To determine whether any theory of gravity has the hope of being renormalizable we look at the generic ultraviolet behavior of L-loop diagrams in d space-time dimensions. First, we note that in all geometrical gravity theories, the propagator and vertex have reciprocal power behavior. For example, in Einstein theory the propagator $\Delta \sim p^{-2}$ and the vertex $V \sim p^2$ in any dimension. Higher derivative terms such as R^2 and R^3 can be added to the Einstein action with somewhat different behavior. Adding an R^2 term introduces p^{-4} dependence into the propagator which improves the UV properties of the theory, however, such a theory is either not unitary or not causal or both. Adding higher powers of R does not affect the propagator but worsens the UV divergences because the vertices contain higher powers of momenta.

Assuming this generic reciprocal behavior, the divergence of a one loop n-point function is proportional to $\int^{\Lambda} d^d p (\Delta V)^n \sim \Lambda^d$. Because of the topological relation

$$L = N_I - N_V + 1 \ , \tag{3.2.1}$$

where N_I and N_V are the number of internal lines and vertices, respectively, higher loops have one more power of propagator than vertex. Each loop also has an additional momentum integral, making the overall divergence $\Lambda^{(L-1)(d-r)}$ times the one loop divergence, where $\Delta \sim V^{-1} \sim p^{-r}$. In order to have a finite

number of counterterms we must have $d - r \leq 0$. Because unitarity forbids $r \geq 4$ and there are no propagating degrees of freedom in a pure gravity theory in $d = 2$, the only possibility is $d = r = 3$. We will see that TMG has this property.

3.3 Topologically Massive Gravity

The action for TMG is given by $S_E + S_{CS}$ where the Einstein and Chern-Simons terms are respectively

$$S_E = \kappa^{-2} \int d^3x \sqrt{-g}\; R \;, \tag{3.3.1a}$$

$$S_{CS} = \frac{1}{2\mu} \int d^3x \;\epsilon^{\mu\nu\lambda} \Gamma^{\rho}_{\lambda\sigma} (\partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho}) \;. \tag{3.3.1b}$$

The field equations are third order in derivatives of the metric, and they are given by

$$\sqrt{-g}\; G^{\mu\nu} + \kappa^2 \mu^{-1} C^{\mu\nu} = 0 \;, \tag{3.3.2}$$

where $G^{\mu\nu}$ and $C^{\mu\nu}$ are the Einstein and Cotton tensors, respectively. Eq (3.3.2) can be split into a trace

$$R = 6\Lambda \;, \tag{3.3.3}$$

and a trace-free part,

$$C^{\mu\nu} = -\mu(R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R) \;. \tag{3.3.4}$$

Just in the case of Einstein gravity the solutions of Eqs (3.3.3) and (3.3.4) are spaces with constant curvature, that is, de Sitter ($\Lambda > 0$), anti-de Sitter ($\Lambda < 0$), or flat ($\Lambda = 0$). But unlike three dimensional Einstein gravity, TMG has a single dynamical mode, a graviton with mass $m = \mu\kappa^{-2}$.

If, as usual, one expands the metric about the flat background,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \;, \tag{3.3.5}$$

one finds that the h propagator has both p^{-3} and p^{-2} components and one cannot apply the simple power counting arguments discussed in the previous section. Instead, following Deser and Yang,[(1)] we parametrize the metric according to

$$g_{\mu\nu} = \kappa^4\Phi^4(\eta_{\mu\nu} + \sqrt{\mu}\ h_{\mu\nu}) = \kappa^4\Phi^4\bar{g}_{\mu\nu}\ . \tag{3.3.6}$$

where h satisfies $h^{\mu}_{\mu} = 0$.[(2)] The action becomes

$$S = \int d^3x\sqrt{-\bar{g}}\ [8\bar{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \Phi^2 R(\bar{g})] + (2\sqrt{\mu})^{-1}S_{CS}(\bar{g})\ . \tag{3.3.7}$$

Since the Chern-Simons action is conformally invariant, it is unaltered by this rescaling. Gauge-fixing is performed by setting $h^{\mu\nu}{}_{,\mu} \equiv \partial_\mu h^{\mu\nu} = 0$; this necessitates the introduction of a Lagrange multiplier B_μ and ghosts b_μ, c^μ. The resulting gauge-fixed action is $S_{GF} = S + S_F + S_G$, where

$$S_F = \int d^3x\ B_\mu\partial_\nu h^{\mu\nu}\ , \tag{3.3.8}$$

is the gauge-fixing term and

$$S_G = -\int d^3x\ b^\mu\partial_\mu\left[\bar{D}^\mu c^\nu + \bar{D}^\nu c^\mu - \frac{2}{3}\bar{g}^{\mu\nu}(\eta_{\alpha\beta}\bar{g}^{\alpha\sigma}\bar{D}_\sigma c^\beta)\right]\ , \tag{3.3.9}$$

is the ghost action. Here $\bar{D}$ is the covariant derivative with respect to $\bar{g}$. The total action is then invariant under the BRST transformation

$$\delta\Phi = \frac{1}{6}\Phi(c^\nu{}_{,\nu} + h^{\lambda\nu}c_{\lambda,\nu}) - c^\lambda\Phi_{,\lambda}\ , \tag{3.3.10a}$$

$$\begin{aligned}\delta h^{\mu\nu} &= \bar{D}^\mu c^\nu + \bar{D}^\nu c^\mu - \frac{2}{3}\bar{g}^{\mu\nu}(\eta_{\alpha\beta}\bar{g}^{\alpha\sigma}\bar{D}_\sigma c^\beta)\\ &= c^{\nu,\mu} + c^{\mu,\nu} + h^{\mu\lambda}c^\nu{}_{,\lambda} + h^{\nu\lambda}c^\mu{}_{,\lambda} - h^{\mu\nu,\lambda}c_\lambda \\ &\quad - \frac{2}{3}(\eta^{\mu\nu} + h^{\mu\nu})(c^\lambda{}_{,\lambda} + h^{\alpha\beta}c_{\alpha,\beta})\ ,\end{aligned} \tag{3.3.10b}$$

(1) Actually Deser and Yang's Φ is ours times a factor of κ.

(2) Note that the indices of h, and all other covariant quantities hereafter, are raised and lowered using the background metric $\eta_{\mu\nu} = (-1, +1, +1)$.

$$\delta c^\alpha = c^{\alpha,\beta} c_\beta \ , \tag{3.3.10c}$$

$$\delta b_\alpha = -B_\alpha \ , \tag{3.3.10d}$$

$$\delta B_\alpha = 0 \ . \tag{3.3.10e}$$

The propagators are

$$< h^{\mu\nu} h^{\alpha\beta} > = \frac{1}{4} p^{-4} p_\gamma (\epsilon^{\mu\alpha\gamma} P^{\nu\beta} + \epsilon^{\nu\alpha\gamma} P^{\mu\beta} + \epsilon^{\mu\beta\gamma} P^{\nu\alpha} + \epsilon^{\nu\beta\gamma} P^{\mu\alpha}) \ , \tag{3.3.11a}$$

$$< \Phi\Phi > = i \frac{1}{16} p^{-2} \ , \tag{3.3.11b}$$

$$< b_\mu c_\nu > = i p^{-2} (\eta_{\mu\nu} - \frac{1}{4} p^{-2} p_\mu p_\nu) \ , \tag{3.3.11c}$$

where $P^{\mu\nu} = \eta^{\mu\nu} - p^{-2} p^\mu p^\nu$. Thus with the parametrization (3.3.6), the h propagator has the desired p^{-3} behavior. The vertices may be found by expanding the action to any desired order (see Appendix A) and they include terms of the form $\Phi^2 \partial^2 (\sqrt{\mu} h)^{n-1}$, $\partial^3 (\sqrt{\mu} h)^{n+1}$ and $b\partial^2 (c\sqrt{\mu} h^{n-1})$ where $n \geq 2$. Note that the vertices from the Einstein term contain exactly two Φ fields, while the other vertices contain none.

We now notice that negative powers of m (equivalently, positive powers of κ) may never appear in any Feynman diagram, since they do not appear in any vertices or propagators, hence κ is a super-renormalizable coupling. The true expansion parameter μ is dimensionless, providing the first indication that the theory may be renormalizable. To prove the naive power-counting argument we determine the highest degree of divergence D of any L-loop diagram in the theory. Let N_I^x and N_V^i be the number of internal lines of species x and the number of vertices of degree ∂^i, respectively. The degree of divergence is then

$$D = 3L + \sum_i i N_V^i - 2N_I^{gh} - 2N_I^\Phi - 3N_I^h \ . \tag{3.3.12}$$

Using (3.2.1) in (3.3.12) one obtains

$$D = 3 - (N_V^{gh} - N_I^{gh}) - (N_V^{\Phi} - N_I^{\Phi}) \leq 3 \ , \tag{3.3.13}$$

where N_V^{gh} and N_V^{Φ} are the number of ghost and Φ vertices respectively. Since any vertex has at most two ghost or two Φ fields, the terms in parentheses are nonnegative and thus the degree of UV divergence is always ≤ 3. These divergences can be absorbed into the coefficients of the Einstein term of dimension -1, the Chern-Simons term of dimension 0 and possibly a cosmological term of dimension -3, thus the theory is power-counting renormalizable.

The loophole in the above argument is the assumption that gauge invariance may be maintained in a regulated version of the theory without giving up the desirable power counting behavior mentioned above. Without the use of such a gauge invariant regulator, there is the possibility that additional terms might be required to cancel the gauge transformation of the effective action, and that these terms might contain negative powers of m. To see if quantum corrections to the theory violate the BRST invariance through such a term, we look for nontrivial solutions to the BRST cohomology problem as follows. Let $\Delta = Q\Gamma$ be the possible violation of BRST symmetry, where Q is the BRST transformation and Γ is the effective action to some loop order. We consider general solutions to the cohomology problem

$$Q\Delta = 0 \ , \qquad Q^2 = 0 \ . \tag{3.3.14}$$

If the solution is trivial, *i.e.* $\Delta = Q\Gamma'$, we can add Γ' as a counterterm to Γ to cancel the anomaly. If the solution is nontrivial and Δ is indeed generated by the quantum corrections then the theory is anomalous. Deser and Yang

determine, through such an analysis, that there is one such possibility: If a term

$$\int c^{\alpha}{}_{,\alpha}\sqrt{-g} = \int c^{\alpha}{}_{,\alpha}\ \Phi^6 + \text{higher order terms}, \tag{3.3.15}$$

arises from the BRST transformation of the effective action, then the counterterm necessary to cancel this term will add negative powers of the mass and hence ruin renormalizability. Deser and Yang showed that to one loop in dimensional regularization this term does not appear. Unfortunately, since the β function for μ vanishes to one loop, one cannot apply the Adler-Bardeen theorems [29,30,31] to conclude that it cannot occur at higher loops. Thus, to determine whether or not this term arises, one must use a suitable gauge invariant regularization.

3.4 Nonlocal Regularization

In this section we review the method of nonlocal regularization. Details may be found in ref. 11. Consider a generic action in d space-time dimensions which can be written as a free part plus an interacting part:

$$S[\phi] = \frac{1}{2}\int d^d x\ \phi_i F_{ij}\phi_j + I[\phi]\ . \tag{3.4.1}$$

where ϕ_i are fields of any type, and F_{ij} of course contains derivatives. We define the nonlocal smearing operator

$$\mathcal{E} \equiv \exp\left(\frac{F}{\Lambda^2}\right)\ , \tag{3.4.2}$$

where Λ is the regularization parameter. The local limit is obtained by taking the $\Lambda \to \infty$ limit. Our convention is that the derivatives in an $\mathcal{E}^2$ act on everything to the right, unless otherwise specified.

For each field ϕ_i, we introduce an auxiliary field ψ_i of the same type, and construct the regulated action

$$\hat{S}[\phi, \psi] = \frac{1}{2} \int d^d x \left(\phi_i \frac{F_{ij}}{\mathcal{E}^2} \phi_j - \psi_i \mathcal{O}_{ij}^{-1} \psi_j \right) + I[\phi + \psi] \ , \tag{3.4.3}$$

where

$$\mathcal{O}_{ij} = \frac{(\mathcal{E}^2 - 1)}{F_{ij}} \ . \tag{3.4.4}$$

It is to be understood that ψ_i are auxiliary fields which are to be eliminated using their equations of motion:

$$\frac{\delta S}{\delta \psi_i} = 0 \ . \tag{3.4.5}$$

Multiplying (3.4.5) by $\mathcal{O}$, we obtain the unique solution for ψ as a functional of ϕ:

$$\psi_i[\phi] = \mathcal{O}_{ij} \frac{\delta I[\phi + \psi]}{\delta \psi_j} \ . \tag{3.4.6}$$

Equation (3.4.6) can be solved iteratively for ψ_i. The solution for ψ_i has a convenient graphical expression: ψ_i is given by evaluating tree amplitudes of the unregulated theory, with a factor of $\mathcal{E}^2 - 1$ on each propagator (see ref. 11 for details). Substituting this solution into (3.4.3) gives the nonlocalized action for the ϕ fields,

$$\hat{S}[\phi] = \hat{S}[\phi, \psi(\phi)] \ . \tag{3.4.7}$$

Suppose that $S[\phi]$ is invariant under any symmetry

$$\delta \phi_i = T_i[\phi] \ . \tag{3.4.8}$$

Let T consist of a linear part plus a nonlinear part, $T = T^l + T^{nl}$, then $\hat{S}[\phi, \psi]$ as defined by (3.4.3) will be invariant under the new symmetry

$$\hat{\delta} \phi_i = T_i^l[\phi] + \mathcal{E}^2 T_i^{nl}[\phi + \psi] \ , \tag{3.4.9a}$$

$$\begin{aligned} \hat{\delta} \psi_i = {} & T_i^l[\psi] + (1 - \mathcal{E}^2) T_i^{nl}[\phi + \psi] \\ & - K_{ij}\Big[\phi + \psi[\phi]\Big] \frac{\delta T_k}{\delta \phi_j} \Big[\phi + \psi[\phi]\Big] \mathcal{E}_{kl}^2 \frac{\delta \hat{S}[\phi]}{\delta \phi_l} \ , \end{aligned} \tag{3.4.9b}$$

where

$$K_{ij}^{-1} \equiv \mathcal{O}_{ij}^{-1} - \frac{\delta^2 I[\phi]}{\delta\phi_i \delta\phi_j} \; . \tag{3.4.9c}$$

In order to obtain (3.4.9b) one must use the equation of motion (3.4.6) for the ψ field. Note that the nonlocalized symmetry transformations can be chosen such that the linear part is independent of the auxiliary fields. Generally, $\hat{S}[\phi, \psi]$ must be gauge-fixed in order that we may solve for $\psi[\phi]$; then the symmetry T represents the BRS symmetry of the gauge-fixed theory.

Classically, the nonlocal action $\hat{S}[\phi]$ is equivalent to the original $S[\phi]$, the former being obtainable from the latter by some field redefinition. The difference arises upon quantization. The old functional measure does not exist in the new basis due to ultraviolet divergences. To quantize the theory a new measure must be constructed which is well-defined in the new basis, is analytic in the momenta, and obeys the symmetries of the theory. The invariance of the quantum theory under the nonlocalized symmetry requires the invariance of the functional integral

$$Z_{\mathcal{E}} \equiv \int [D\phi]\mu[\phi] \Big(\text{Gauge fixing}\Big) \exp\Big(i\hat{S}[\phi]\Big) \; . \tag{3.4.10}$$

Although the full action including gauge-fixing terms is invariant under the symmetry transformations, a measure factor

$$\mu[\phi] = e^{iS_M[\phi]} \; , \tag{3.4.11}$$

must be introduced to insure invariance of the functional measure:

$$\hat{\delta}\Big([D\phi]\mu[\phi]\Big) = 0 \; . \tag{3.4.12}$$

The condition of Eq. (3.4.12) relates the variation of the measure factor to the Jacobian of the tranformation via

$$
\begin{aligned}
\hat{\delta} S_M[\phi] &= -\operatorname{Tr}\left\{\frac{\delta \hat{\delta}\phi_i}{\delta \phi_m}\right\} \\
&= -\operatorname{Tr}\left\{\mathcal{E}^2_{ij}\frac{\delta T_j}{\delta \phi_k}\Big[\phi+\psi[\phi]\Big]\mathcal{O}^{-1}_{kl}K_{lm}\Big[\phi+\psi[\phi]\Big]\right\} ,
\end{aligned}
\tag{3.4.13}
$$

where the second equality uses (3.4.9) and the trace is over space-time coordinates. We can use (3.4.13) to solve for the measure factor order by order, resulting in a completely invariant theory. In practice, this is difficult to do for higher order terms, and it is hoped that further study of nonlocal theories will reveal easier ways of generating the measure factor. We must also note that it has not yet been proven that it is always possible to construct an appropriate measure factor to all orders. If such a measure factor does not exist for the theory then a local symmetry is potentially anomalous. For our arguments concerning TMG, we will be assuming that an appropriate measure does exist.

3.5 Nonlocal Feynman Rules

We have described how to obtain the nonlocal action (3.4.7) by solving the auxiliary field equation of motion. However the Feynman rules for Green's functions in this theory are inconvenient for calculations due to all of the interactions induced when the auxiliary field is eliminated. Instead we will work with the Feynman rules derived from the action $S[\phi,\psi]$ which are closer to those of the original theory and enforce the condition that ψ_i satisfy its classical field equation by requiring that there are no closed loops consisting of only ψ lines (ψ must be on-shell in any diagram). Since one is interested in amplitudes involving the physical field ϕ, no auxiliary fields appear as external lines.

The general Feynman rules in the theory in terms of $S[\phi,\psi]$ are as follows. The ϕ and ψ propagators are

$$\frac{i\mathcal{E}^2}{F+i\epsilon} = -i\int_1^\infty \frac{d\tau}{\Lambda^2}\exp\Bigl(\tau\frac{F}{\Lambda^2}\Bigr) , \tag{3.5.1}$$

$$-i\mathcal{O} = \frac{i(1-\mathcal{E}^2)}{F} = -i\int_0^1 \frac{d\tau}{\Lambda^2}\exp\Bigl(\tau\frac{F}{\Lambda^2}\Bigr) , \tag{3.5.2}$$

respectively. The ψ field is indeed an auxiliary field which should not appear on any external legs, as its propagator, from Eq. (3.5.2), has no pole. The vertices are of the same form as in the local theory. The higher induced vertices in the $\hat{S}[\phi]$ theory are obtained graphically from the $S[\phi,\psi]$ theory: they are the connected tree diagrams which follow from using the local interaction vertices but with propagators replaced by $-i\mathcal{O}$ (ψ lines). There are also vertices from the measure factor which will be connected only to ψ lines. Questions such as reducibility of Feynman diagrams in the theory in terms of $S[\phi,\psi]$ are resolved as in the $\hat{S}[\phi]$ theory with the additional requirement that ψ lines cannot be cut. Feynman rules for nonlocal TMG are collected in Appendix A.

We have shown that there should exist an appropriate measure factor in order to have a well-defined anomaly free quantum theory. In most cases it is extremely difficult construct it, however it can be computed perturbatively to any order in the coupling of the theory. If we expand Eq. (3.4.13) we can obtain a set of Feynman rules for calculating the variation of the measure factor under the nonlocal symmetry. Writing

$$\begin{aligned} K_{ij} &= \mathcal{O}_{ik}\Bigl(\delta_{kj} - \mathcal{O}_{kl}\frac{\delta^2 I}{\delta\phi_l\delta\phi_j}\Bigr)^{-1} \\ &= \mathcal{O}_{ik}\Bigl(\delta_{kj} + \mathcal{O}_{kl}\frac{\delta^2 I}{\delta\phi_l\delta\phi_j} + \mathcal{O}_{kl}\frac{\delta^2 I}{\delta\phi_l\delta\phi_m}\mathcal{O}_{mn}\frac{\delta^2 I}{\delta\phi_n\delta\phi_j} + \dots\Bigr) , \end{aligned} \tag{3.5.3}$$

and inserting this in (3.4.13), one obtains

$$\hat{\delta} S_M[\phi] = -\mathrm{Tr}\Big\{ \mathcal{E}^2_{ij} \frac{\delta T_j}{\delta \phi_k}\Big[\phi + \psi[\phi]\Big]\Big(\delta_{km} + \mathcal{O}_{kl}\frac{\delta^2 I}{\delta\phi_l \delta\phi_m} + \mathcal{O}_{kl}\frac{\delta^2 I}{\delta\phi_l\delta\phi_n}\mathcal{O}_{np}\frac{\delta^2 I}{\delta\phi_p\delta\phi_m} + \ldots\Big)\Big[\phi+\psi[\phi]\Big]\Big\}\,. \tag{3.5.4}$$

We may read off diagrammatic rules from this expression by writing it in momentum space. Since there is only one trace over space-time coordinates, we need only look at one loop diagrams. Each diagram has a single vertex factor coming from $\frac{\delta T}{\delta \phi}$. The remaining vertices are arbitrary in number and are the same as those discussed above. The first type of vertex always connects to an internal line with "propagator" $\mathcal{E}^2$. The other vertices connect either to two internal lines with "propagator" $\mathcal{O}$ or to one internal line of each type. The external legs correspond to either ϕ or ψ fields. These diagrammatic rules for TMG are given in Appendix B. By computing all one loop n-point diagrams of this type one obtains a perturbative expression for $\hat{\delta} S_M$ which must be inverted to get the measure factor S_M.

3.6 Renormalizability

We now apply this method to TMG. We associate auxiliary fields Ψ, $k_{\mu\nu}$, d^μ with the fields Φ, $h_{\mu\nu}$, c^μ respectively. In this field basis each field is massless, so the smearing operator is simply

$$\mathcal{E}_0^2 \equiv \exp\left(\partial^2/\Lambda^2\right)\ . \tag{3.6.1}$$

The gauge transformation laws for the fields then become nonlocalized according to (3.4.9a):

$$\delta\Phi = \mathcal{E}_0^2\Big[\frac{1}{6}(\Phi+\Psi)\Big((c+d)^\nu{}_{,\nu} + (h+k)^{\mu\nu}(c+d)_{\mu,\nu}\Big) - (c+d)^\lambda(\Phi+\Psi)_{,\lambda}\Big]\ , \tag{3.6.2a}$$

$$\delta h^{\mu\nu} = c^{\nu,\mu} + c^{\mu,\nu} - \frac{2}{3}\eta^{\mu\nu}c^{\alpha}{}_{,\alpha} + \mathcal{E}_0^2\Big[(h+k)^{\mu\lambda}(c+d)^{\nu}{}_{,\lambda}$$
$$+ (h+k)^{\nu\lambda}(c+d)^{\mu}{}_{,\lambda} - (c+d)^{\lambda}(h+k)^{\mu\nu}{}_{,\lambda}$$
$$- \frac{2}{3}\eta^{\mu\nu}(h+k)_{\alpha\beta}(c+d)^{\alpha,\beta} \tag{3.6.2b}$$
$$- \frac{2}{3}(h+k)^{\mu\nu}(c+d)^{\alpha}{}_{,\alpha}$$
$$- \frac{2}{3}(h+k)^{\mu\nu}(h+k)_{\alpha\beta}(c+d)^{\alpha,\beta}\Big] ,$$

$$\delta c^{\alpha} = \mathcal{E}_0^2(c+d)^{\alpha,\beta}(c+d)_{\beta} , \tag{3.6.2c}$$

$$\delta b_{\alpha} = -B_{\alpha} , \tag{3.6.2d}$$

$$\delta B_{\alpha} = 0 . \tag{3.6.2e}$$

We see that nonlocal regularization gives a regulated theory which is automatically BRS invariant, but it must be checked that the desirable power-counting behavior of the unregulated theory still persists. Negative powers of m could be generated either in the measure factor, or by the loop integrations themselves. We will examine each of these possibilities.

We now examine (3.4.13) to determine which terms could possibly give a contribution of the form (3.3.15). Tr $\frac{\delta(\delta\Phi)}{\delta\Phi}$ cannot contribute to this term. This is because Φ and b, c each couple only to h, so every tree graph which contributes to Ψ or d^{μ} must include at least one h. By the same token, each term in $k^{\mu\nu}$ includes either an h, a pair of ghosts, or a pair of Φ's with derivatives on them, none of which is what we are looking for. So the Φ term does not contribute to the possible anomaly. The c ghost term also does not contribute, for the same reasons. Since b and B do not contribute to the measure factor at all, the only possible contribution is from Tr $\frac{\delta(\delta h^{\mu\nu})}{\delta h^{\mu\nu}}$. We find from (3.6.1b), that this contribution comes from the diagram shown in Figure 1. Since its evaluation involves the product of three ϵ tensors contracted with a term of the form $\Phi^6\partial^{\alpha}c^{\beta}$, an odd number of indices will be left over, with nothing to be

contracted into after the internal momentum has been integrated over. This is necessarily zero by Lorentz invariance. Therefore there are no contributions to the measure factor of the form (3.3.15).

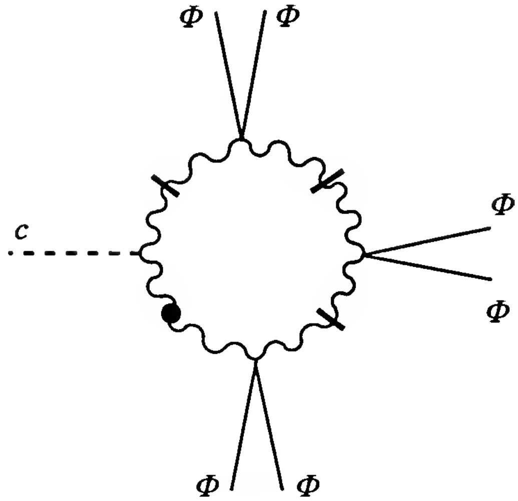


Figure 1. Contribution to the measure factor of the form of Eq. (3.3.15). Here a wavy line with a bar corresponds to the k propagator where k is the auxiliary field field for h. A wavy line with a dot corresponds to a "propagator" given by the smearing operator $\mathcal{E}_0^2$ as described in Sect. 3.5.

In the remaining we show that such powers do not arise from the loop integrations themselves. This situation would correspond to divergences in the limit where $m \to 0$, but Λ remains finite. Since nonlocal regularization regulates all integrals at $p \to \infty$, unregulated divergences can only occur for $p \to 0$, it i.e. in the infrared, which is not at all affected by this procedure.

In contrast to ultraviolet divergences, which are determined by the net effect of all the propagators around an entire loop, infrared singularities are determined by a single propagator, or any group of propagators, whose momentum goes to zero. If we let all the loop momenta be independent and keep momentum-conserving delta functions on the vertices, then we see that the h propagator cannot cause power law divergences for $m, p \to 0$, and the Φ and

ghost propagators actually help matters because they go like $\frac{d^3p}{p^2}$. The only possible divergences of this type will be when one or more of the momentum-conserving delta functions give $\delta^3(0)$. This can happen if all of the momenta going into the vertex go to zero. However, each vertex contains derivatives, which will in momentum space give powers of the momenta which will necessarily soften these singularities. Specifically, each vertex either contains three derivatives, which softens the singularity to at most logarithmic, or two derivatives plus Φ or ghost propagators or positive powers of m, or both. In any case there is no possibility of a power law singularity for $m \to 0$.

In higher loops, there is the possibility that a $\delta(0)$ singularity might be generated not by one vertex but by a combination of two or more. This can happen when some subgraph is imbedded inside another graph. For $m \to 0$ this graph may be singular when the momentum on the line connecting the subgraph to the other graph is zero. If the momentum factors associated with the vertices lie on the subgraph then they will not cancel the singularity. In this case we must argue by induction against the possibility of any problem being caused. Suppose that the theory is proved to be renormalizable to $N-1$ loops, and that this renormalization has been carried out. Then consider an N loop diagram of the type described above. The subgraph of this graph has fewer than N loops, so by assumption the theory must at this point contain counterterms to make this subgraph finite. But by simple dimensional analysis, the subgraph must have dimension $3 - n/2$, where n is the number of Φ or ghost (not h) lines coming out of the subgraph. Since the subgraph plus counterterms must be finite, the graph cannot achieve this dimension by being proportional to the regulator Λ. The only other relevant dimensionful quantities are m and the momenta coming out of the subgraph. These quantities all necessarily soften

the singularity from the delta functions to be at most logarithmic. Note that logarithmic infrared singularities are not a problem as they merely indicate the presence of $\ln \Lambda/m$ terms.

We therefore see that infrared power-law divergences do not arise as $m \to 0$, so that negative powers of m do not arise from loop integrations. Thus, assuming that a gauge invariant measure exists for the nonlocal theory, the theory is indeed anomaly free and hence renormalizable. Our result will still hold if this assumption is false, if the violation of gauge invariance is such that the noninvariant terms in the effective action vanish to all loop orders in the local limit, $\Lambda \to \infty$. Since TMG has no actual gauge anomalies, this is a reasonable assumption, but is by no means a foregone conclusion. Thus at present we have discussed but one approach which gives strong support to the conjecture of Deser and Yang. Our result cannot be considered a proof until the existence of the appropriate measure factor is established.

CHAPTER 4
THERMO-FIELD DYNAMICS OF BLACK HOLES

4.1 Introduction

It is known that the quantization of fields on spacetimes with causally disconnected regions leads to particle creation from the vacuum as a consequence of the information loss associated with the presence of the event horizon(s) [12]. In stationary spacetimes with a simply connected event horizon (such as a stationary black hole, an accelerated observer in Minkowski spacetime, or de Sitter type cosmologies), the emitted particles have a thermal spectrum [13]. This result has been first obtained by Hawking [14] for black holes. He has shown, that the effective temperature of this radiation is $T_H = \frac{\kappa}{2\pi}$, the Hawking temperature, where κ is the surface gravity of the black hole (units are chosen throughout such that k $= \hbar =$ c $= G = 1$). These results have been confirmed and derived in a number of ways, and several attempts have been made to gain a better understanding of the features of the Hawking process and the physical role of the event horizon [14,15,16]. A particularly interesting approach into this direction is the one used by Israel [16], who considered the problem of particle creation on the Schwarzschild background. The idea is to quantize the fields in the full analytically extended Schwarzschild spacetime (known as the Kruskal extension) in order to keep track of particle states on the hidden side of the horizon as well. The same idea allows us to apply a quantum-statistical formalism, known as thermo-field dynamics [17].

Here we briefly review the canonical formulation of thermo-field dynamics for free fields. This will provide the main ideas and all the technical tools we need (for more general and detailed discussion see *e.g.*, ref. 17). The central idea of thermo-field dynamics is to express the statistical average of any operator O as a single vacuum expectation value

$$\langle O \rangle = \langle 0(\beta) |\, O \,| 0(\beta) \rangle \ , \tag{4.1.1}$$

where β is the inverse temperature. This can be achieved by augmenting the physical Fock space $\mathcal{F}$ by a fictitious, dual Fock space $\tilde{\mathcal{F}}$. That is, for each operator $O(a^\dagger_{\omega j}, a_{\omega j})$ and each state vector $|\, n\rangle = \prod_{\omega,j} \frac{1}{\sqrt{n_{\omega j}!}} a^{n_{\omega j}} \,|\, 0\rangle$ we introduce a dual operator $\tilde{O}(\tilde{a}^\dagger_{\omega j}, \tilde{a}_{\omega j})$ and a dual state vector $|\, \tilde{n}\rangle = \prod_{\omega,j} \frac{1}{\sqrt{\tilde{n}_{\omega j}!}} \tilde{a}_{\omega j}{}^{\tilde{n}_{\omega j}} \,|\, \tilde{0}\rangle$, where $a^\dagger_{\omega j}$, $\tilde{a}^\dagger_{\omega j}$, $a_{\omega j}$, and $\tilde{a}_{\omega j}$ are creation and annihilation operators of the ωj modes (j labels the degeneracy of the energy level ω) with the usual commutation (anticommutation) relations. Namely, for bosons the only nonzero commutators, and for fermions the only nonzero anticommutators are

$$[a^\dagger_{\omega j}, a_{\omega' j'}] = [\tilde{a}^\dagger_{\omega j}, \tilde{a}_{\omega' j'}] = \delta_{jj'}\delta(\omega - \omega') \ , \tag{4.1.2}$$

and

$$\{a^\dagger_{\omega j}, a_{\omega' j'}\} = \{\tilde{a}^\dagger_{\omega j}, \tilde{a}_{\omega' j'}\} = \delta_{jj'}\delta(\omega - \omega') \ , \tag{4.1.3}$$

respectively. The states $|\, 0\rangle$, and $|\, \tilde{0}\rangle$ are the vacuum states annihilated by $a_{\omega j}$ and $\tilde{a}_{\omega j}$ respectively.

In the direct product Fock space $\mathcal{F} \otimes \tilde{\mathcal{F}}$, spanned by the state vectors $|\, n, \tilde{m}\rangle = |\, n\rangle \otimes |\, \tilde{m}\rangle$, the temperature dependent vacuum state $|\, 0(\beta)\rangle$ in (4.1.1) is given by a Bogoliubov transformation of $|\, 0, \tilde{0}\rangle$

$$\begin{aligned} |\, 0(\beta)\rangle &= e^{iG(\theta)} \,|\, 0, \tilde{0}\rangle \ , \\ G(\theta) &= -i \sum_{\omega j} \theta_\omega (\tilde{a}_{\omega j} a_{\omega j} - \tilde{a}^\dagger_{\omega j} a^\dagger_{\omega j}) \ , \end{aligned} \tag{4.1.4}$$

with the θ_ω parameters defined by

$$\begin{aligned} \sinh^2\theta_\omega &= \frac{1}{e^{\omega\beta}-1} \qquad \text{for bosons,} \\ \sin^2\theta_\omega &= \frac{1}{e^{\omega\beta}+1} \qquad \text{for fermions.} \end{aligned} \tag{4.1.5}$$

We also introduce the operators

$$\begin{aligned} a^\dagger_{\omega j}(\beta) &= e^{-iG}\, a^\dagger_{\omega j}\, e^{iG}, \qquad & a_{\omega j}(\beta) &= e^{-iG}\, a_{\omega j}\, e^{iG} \\ \tilde{a}^\dagger_{\omega j}(\beta) &= e^{-iG}\, \tilde{a}^\dagger_{\omega j}\, e^{iG}, \qquad & \tilde{a}_{\omega j}(\beta) &= e^{-iG}\, \tilde{a}_{\omega j}\, e^{iG}, \end{aligned}$$

such that they satisfy the same commutation (anticommutation) relations (4.1.2) and (4.1.3). The state $|\,0(\beta)\rangle$ is annihilated by the operators $a_{\omega j}(\beta)$ and $\tilde{a}_{\omega j}(\beta)$, and the entire Fock space can be constructed successively from $|\,0(\beta)\rangle$ using the creation operators $a^\dagger_{\omega j}(\beta)$ and $\tilde{a}^\dagger_{\omega j}(\beta)$.

Using the above construction of $|\,0(\beta)\rangle$ the statistical average of any physical operator (a functional of $a_{\omega j}$ and $a^\dagger_{\omega j}$ only) can be expressed as a vacuum expectation value of the form (4.1.1). In particular, as it is easily seen from Eqs. (4.1.4) and (4.1.5), the average number of the ωj modes are given by the familiar Fermi and Bose distributions.

The formalism described above can easily be generalized to black holes by identifying the physical Fock space $\mathcal{F}$ with particle states outside the horizon, and the tilde space $\tilde{\mathcal{F}}$ with particle states inside the horizon. The above decomposition of the whole Fock space into the direct product space $\mathcal{F}\otimes\tilde{\mathcal{F}}$ corresponds to the conventional definition of positive frequency modes (particle states). It is known [13,14,15,16] that this definition with respect to the Schwarzschild time coordinate t (which is defined by a timelike Killing vector field $\partial/\partial t$ everywhere outside the horizon) leads to a mode expansion of the fields which is not analytic on the horizon. On the other hand a linear combination of these "Schwarzschild" modes can be found, which leads to an analytic

mode expansion. Note that the latter corresponds to positive frequency modes with respect to the Kruskal time coordinate, which defines a Killing vector field only on the horizon. This linear combination is generated by a Bogoliubov transformation, which is uniquely determined by the requirement that the fields be analytic on the horizon [15,16]. The formalism described above can be applied to the problem of particle creation by black holes, and as we shall see, far from the black hole it leads to a thermal distribution with the Hawking temperature.

We begin by reviewing of Israel's paper [16]. The quantization of a massless scalar field on the Schwarzschild background is considered by using thermofield dynamics. We conclude that the results obtained this way are equivalent to the earlier ones. In the rest of the paper we describe the possible generalizations of this approach. In Sect. 4.3 an approximate multi-black hole solution is considered as an example to demonstrate how to extend the method to spacetimes with many causally disconnected regions. Finally in Sect. 4.4 we derive the Hawking radiation of a black hole emitting neutrinos and antineutrinos.

4.2 Massless Scalar Particles on the Schwarzschild Background

We first consider the creation of massless scalar particles in the gravitational field of a black hole. We consider the Schwarzschild metric

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2 d\Omega^2 \ , \tag{4.2.1}$$

as an example, and look for solutions of the massless scalar field equation,

$$\sqrt{-g}(\Phi_{,\mu}\, g^{\mu\nu}(-g)^{-\frac{1}{2}})_{,\nu} = 0 \ , \tag{4.2.2}$$

as the superposition of a complete set of positive frequency modes of the form:

$$\Phi_{\omega lm}(r, \vartheta, \varphi) = \frac{1}{\sqrt{2\pi|\omega|}} f_{\omega l}(r) Y_{lm}(\vartheta, \varphi) \ . \tag{4.2.3}$$

Here $f_{\omega l}$ obeys the equation:

$$\Big(\frac{1}{r^2}\frac{d}{dr^*}r^2\frac{d}{dr^*}+\omega^2-(1-\frac{2M}{r})\frac{l(l+1)}{r^2}\Big)f_{\omega l}(r)=0\ , \tag{4.2.4}$$

and

$$r^*=r+2M\ln\Big|\frac{r}{2M}-1\Big|\ . \tag{4.2.5}$$

Let us consider the solutions of Eq. (4.2.4) which correspond to outgoing modes at the past horizon $\mathcal{H}^-$. Near the horizon these solutions behave like

$$f_{\omega lm}\sim e^{-i\omega u}\ , \tag{4.2.6}$$

where $\omega > 0$ and $u = t - r^*$. This is the usual definition of positive frequency states with respect to the Schwarzschild time. Everywhere outside the horizon, $\frac{\partial}{\partial t}$ is a timelike Killing vector.

It is also possible to define positive frequency modes with respect to the Kruskal time coordinate U. In Kruskal coordinates [15],

$$\begin{aligned} U&=-4Me^{-\frac{u}{4M}}, \quad V=4Me^{\frac{v}{4M}},\\ u&=t-r^*, \quad v=t+r^*, \end{aligned} \tag{4.2.7}$$

the Schwarzschild metric has the form

$$ds^2=2Mr^{-1}e^{-\frac{r}{2M}}dUdV-r^2d\Omega^2\ . \tag{4.2.8}$$

On the past horizon $\mathcal{H}^-$, $\partial/\partial U$ is a null Killing vector. Note that $f_{\omega lm}$ defined by Eq. (4.2.6),

$$f_{\omega lm}\sim e^{-i\omega u}\sim e^{-i\omega\ln|U|}\ , \tag{4.2.9}$$

is also a complete set of positive frequency solutions ($\omega > 0$) with respect to the Kruskal time U.

Since $\mathcal{H}^-$ divides spacetime into two causally disconnected regions, the one outside the horizon (region I) and the other inside the horizon (region II), two "Schwarzschild" modes can be associated with any given solution $f_{\omega lm}$:

$$F_\omega^{(+)} = \begin{cases} \frac{1}{\sqrt{2\pi|\omega|}} Y_{lm}(\vartheta,\varphi) e^{-\frac{i\omega}{\kappa}\ln|U|} & \text{outside the horizon} \\ 0 & \text{inside the horizon,} \end{cases} \tag{4.2.10a}$$

$$F_\omega^{(-)} = \begin{cases} 0 & \text{outside the horizon} \\ \frac{1}{\sqrt{2\pi|\omega|}} Y_{lm}(\vartheta,\varphi) e^{+\frac{i\omega}{\kappa}\ln|U|} & \text{inside the horizon.} \end{cases} \tag{4.2.10b}$$

Note however that $F_\omega^{(\pm)}$ are singular because U goes to zero on the future horizon. On the other hand, the following linear combinations:

$$H_\omega^{(+)} = F_\omega^{(+)} \cosh\theta_\omega + F_\omega^{(-)} \sinh\theta_\omega \ , \tag{4.2.11a}$$

$$H_\omega^{(-)} = F_\omega^{(+)} \sinh\theta_\omega + F_\omega^{(-)} \cosh\theta_\omega \ , \tag{4.2.11b}$$

are analytic in the lower half complex U plane if $\tanh\theta_\omega = e^{-\frac{\pi\omega}{\kappa}}$. The modes $H_\omega^{(\pm)}$ are positive (negative) frequency "Kruskal" modes. Both $F_{\omega j}^{(\pm)}$ and $H_{\omega j}^{(\pm)}$ are complete sets of modes, satisfying the orthonormality conditions

$$(F_{\omega j}^{(\pm)}, F_{\omega' j'}^{(\pm)}) = \pm\delta_{jj'}\delta(\omega-\omega') \ , \tag{4.2.12a}$$

$$(H_{\omega j}^{(\pm)}, H_{\omega' j'}^{(\pm)}) = \pm\delta_{jj'}\delta(\omega-\omega') \ , \tag{4.2.12b}$$

with respect to the Klein-Gordon scalar product,

$$(F_1, F_2) = i\int_\Sigma (F_1^*\partial_\mu F_2 - F_2\partial_\mu F_1^*) n^\mu d\Sigma \ , \tag{4.2.13}$$

where n^μ is a future directed unit vector orthogonal to the Cauchy surface Σ and $d\Sigma$ is the volume element in Σ.

To quantize the real scalar field Φ in terms of $F_{\omega j}^{(\pm)}(\omega > 0)$, we expand

$$\Phi = \sum_{\omega,j}\left(a_{\omega j}^{(+)} F_{\omega j}^{(+)} + a_{\omega j}^{(-)\dagger} F_{\omega j}^{(-)} + \text{h.c.}\right) \ , \tag{4.2.14}$$

where $a^{(\pm)\dagger}_{\omega j}$ and $a^{(\pm)}_{\omega j}$ are creation and annihilation operators, respectively, obeying the usual commutation relations, that is the only nonzero commutators are

$$[a^{(\pm)}_{\omega j}, a^{(\pm)\dagger}_{\omega' j'}] = \delta_{jj'}\delta(\omega - \omega') \ . \tag{4.2.15}$$

The alternative expansion of $\Phi(x)$ in terms of $H^{(\pm)}_{\omega j}$ is

$$\Phi = \sum_{\omega,j} \left(a^{(+)}_{\omega j}(\kappa) H^{(+)}_{\omega j} + a^{(-)\dagger}_{\omega j}(\kappa) H^{(-)}_{\omega j} + \text{h.c.} \right) \ . \tag{4.2.16}$$

The operators $a^{(\pm)\dagger}_{\omega j}(\kappa)$ and $a^{(\pm)}_{\omega j}(\kappa)$ also satisfy the commutation relations (4.2.15), and are given by the Bogoliubov transformation:

$$\begin{aligned} a^{(\pm)}_{\omega j}(\kappa) &= \exp(-iG) a^{(\pm)}_{\omega j} \exp(iG) \\ &= \cosh\theta_\omega a^{(\pm)}_{\omega j} - \sinh\theta_\omega a^{(\mp)\dagger}_{\omega j} \ , \end{aligned} \tag{4.2.17}$$

where the hermitian operator G is defined by

$$G = \sum_{\omega,j} i\theta_\omega (a^{(+)\dagger}_{\omega j} a^{(-)\dagger}_{\omega j} - a^{(+)}_{\omega j} a^{(-)}_{\omega j}) \ . \tag{4.2.18}$$

The physical vacuum state near the black hole, the "Kruskal" vacuum, is determined by requiring that freely falling observers encounter no singularities as they pass through the horizon. If $|0\rangle$ denotes the "Schwarzschild" vacuum annihilated by the operators $a^{(\pm)}_{\omega j}$, then the "Kruskal" vacuum:

$$|0(\kappa)\rangle = \exp(-iG)\ |0\rangle \ , \tag{4.2.19}$$

is annihilated by the operators $a^{(\pm)}_{\omega j}(\kappa)$.

Far from the black hole, at infinity, the observable quantities are the vacuum expectation values of the operators of the form $O(a^\dagger_{\omega j}, a_{\omega j})$ calculated in the Kruskal vacuum. In particular the average number of ωj modes at infinity,

$$\langle 0(\kappa)\ | a^\dagger_{\omega j} a_{\omega j}\ | 0(\kappa)\rangle = \sinh^2\theta_\omega = \frac{1}{e^{\frac{2\pi\omega}{\kappa}} - 1} \ , \tag{4.2.20}$$

is given by a thermal distribution with temperature equal to the Hawking temperature T_H.

4.3 Many Black Holes

The method can be extended to spacetimes with more than two causally disconnected regions. Because such a solution is not known, we consider the idealized case of N well-separated black holes as an example. Our assumptions are as follows: (1) they can be considered static; (2) far from the black holes the metric is approximately Minkowski; (3) the radiation emitted by the individual black holes is uncorrelated. By approximation (1) and (2) the solution of the scalar field equation is $f_{\omega lm} \sim e^{-i\omega(t-r)}$ and near the horizon of the i'th black hole the metric is approximately Schwarzschild. Thus, $f_{\omega lm} \sim e^{\mp\frac{i\omega}{\kappa_i}\ln|U_i|}$ where i=1,2...N and κ_i are the surface gravities of the horizons.

With these assumptions a linear combination of the "Schwarzschild" modes can be found which is analytic everywhere. The creation, annihilation operators and the corresponding vacuum state are given by a Boguliubov transformation. The new vacuum state will not appear to be empty for a stationary observer far from the black holes. The spectrum of the emitted particles (in this idealized case) will be the sum of the individual black hole's thermal spectra.

We first consider the N=2 case. The solution of the field equation is such that

$$f_{\omega lm} \sim \begin{cases} e^{\mp i\omega(t-r)} & \text{far from the black holes} \\ e^{\mp\frac{i\omega}{\kappa_i}\ln|U_i|} & \text{at the horizon of the i'th black hole.} \end{cases} \tag{4.3.1}$$

In order to define a complete set of modes we divide the space into two cells, both of them containing only one black hole. The positive frequency modes

are such that $F^{(\pm)}_{i\omega j}$ are nonzero in the i'th cell only, and they are given by

$$F^{(+)}_{i\omega j} = \begin{cases} 0 & \text{inside i'th horizon} \\ \frac{1}{\sqrt{2\pi|\omega|}} Y_{lm}(\vartheta,\varphi) e^{-\frac{i\omega}{\kappa_i}\ln|U_i|} & \text{elsewhere,} \end{cases} \tag{4.3.2}$$

$$F^{(-)}_{i\omega j} = \begin{cases} \frac{1}{\sqrt{2\pi|\omega|}} Y_{lm}(\vartheta,\varphi) e^{+\frac{i\omega}{\kappa_i}\ln|U_i|} & \text{outside i'th horizon} \\ 0 & \text{elsewhere,} \end{cases} \tag{4.3.3}$$

with the normalization,

$$(F^{(\pm)}_{i\omega j}, F^{(\pm)}_{i'\omega' j'}) = \delta_{ii'}\delta_{jj'}\delta(\omega-\omega') \ . \tag{4.3.4}$$

Analogous to the one black hole case we consider another set of modes such that they are given by a linear combination of the above defined "Schwarzschild" modes and are analytic everywhere. These are given by

$$\begin{aligned} H^{(+)}_{1\omega j} &= F^{(+)}_{1\omega j}\cosh\theta_{1\omega}\cos\phi_\omega - F^{(+)}_{2\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \\ &+ F^{(-)}_{1\omega j}\sinh\theta_{1\omega}\cos\phi_\omega - F^{(-)}_{2\omega j}\sinh\theta_{2\omega}\sin\phi_\omega \ , \end{aligned} \tag{4.3.5a}$$

$$\begin{aligned} H^{(+)}_{2\omega j} &= F^{(+)}_{2\omega j}\cosh\theta_{2\omega}\sin\phi_\omega + F^{(+)}_{1\omega j}\cosh\theta_{1\omega}\cos\phi_\omega \\ &+ F^{(-)}_{2\omega j}\sinh\theta_{2\omega}\cos\phi_\omega + F^{(-)}_{1\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \ , \end{aligned} \tag{4.3.5b}$$

$$\begin{aligned} H^{(-)}_{1\omega j} &= F^{(+)}_{1\omega j}\sinh\theta_{1\omega}\cos\phi_\omega - F^{(+)}_{2\omega j}\sinh\theta_{2\omega}\sin\phi_\omega \\ &+ F^{(-)}_{1\omega j}\cosh\theta_{1\omega}\cos\phi_\omega - F^{(-)}_{2\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \ , \end{aligned} \tag{4.3.5c}$$

$$\begin{aligned} H^{(-)}_{2\omega j} &= F^{(+)}_{2\omega j}\sinh\theta_{2\omega}\cos\phi_\omega + F^{(+)}_{1\omega j}\sinh\theta_{1\omega}\sin\phi_\omega \\ &+ F^{(-)}_{2\omega j}\cosh\theta_{1\omega}\cos\phi_\omega + F^{(-)}_{1\omega j}\cosh\theta_{1\omega}\sin\phi_\omega \ . \end{aligned} \tag{4.3.5d}$$

Note first that the solutions are matched on the horizons of the individual black holes independently by choosing the θ_ω parameters such that

$$\frac{H^{(\pm)}_{i\omega j}(\text{region I of the i'th black hole})}{H^{(\pm)}_{i\omega j}(\text{region II of the i'th black hole})} = \tanh\theta_{i\omega} = e^{-\frac{\pi\omega}{\kappa_i}} \ , \tag{4.3.6}$$

which is exactly the analyticity condition for the modes in the case of one black hole. The remaining freedom is used to match the solution between the black holes, that is on the "wall" separating the two cells. To do this we set the ϕ_ω parameters to be

$$\tan\phi_\omega = \frac{\cosh\theta_{1\omega}}{\cosh\theta_{2\omega}} = \left(\frac{1-\exp(-\frac{2\pi\omega}{\kappa_2})}{1-\exp(-\frac{2\pi\omega}{\kappa_1})}\right)^{1/2} . \tag{4.3.7}$$

$H^{(\pm)}_{i\omega j}$ are also normalized according to

$$(H^{(\pm)}_{i\omega j}, H^{(\pm)}_{i'\omega' j'}) = \pm\delta_{ii'}\delta_{jj'}\delta(\omega-\omega') . \tag{4.3.8}$$

To quantize we expand in terms of $F^{(\pm)}_{i\omega j}$

$$\Phi = \sum_{i,\omega,j}\left(F^{(+)}_{i\omega j}a^{(+)}_{i\omega j} + F^{(-)}_{i\omega j}a^{(-)\dagger}_{i\omega j} + \text{h.c.}\right) , \tag{4.3.9}$$

where $a^{(\pm)}_{i\omega j}$ satisfy the commutation relations

$$[a^{(\pm)}_{i\omega j}, a^{(\pm)\dagger}_{i'\omega' j'}] = \delta_{i,i'}\delta_{j,j'}\delta(\omega-\omega') , \tag{4.3.10}$$

the other commutators being zero. The vacuum state is defined by

$$a^{(\pm)}_{i\omega j}|0\rangle = 0 . \tag{4.3.11}$$

But we can also expand in terms of $H^{(\pm)}_{i\omega j}$:

$$\Phi = \sum_{i,\omega,j}\left(H^{(+)}_{i\omega j}a^{(+)}_{i\omega j}(\kappa_i) + H^{(-)}_{i\omega j}a^{(-)\dagger}_{i\omega j}(\kappa_i) + \text{h.c}\right) , \tag{4.3.12}$$

where $a^{(\pm)}_{i\omega j}(\kappa)$ also satisfy the commutation relations,

$$[a^{(\pm)}_{i\omega j}(\kappa), a^{(\pm)\dagger}_{i'\omega' j'}(\kappa)] = \delta_{ii'}\delta_{jj'}\delta(\omega-\omega') , \tag{4.3.13}$$

the other commutators being zero. The vacuum state is defined by

$$a^{(\pm)}_{i\omega j}(\kappa)\,|0(\kappa)\rangle = 0 . \tag{4.3.14}$$

The operators $a^{(\pm)}_{i\omega j}(\kappa)$ are given by the Bogoliubov transformation

$$\begin{aligned} a^{(+)}_{1\omega j}(\kappa) = a^{(+)}_{1\omega j} \cosh\theta_{1\omega}\cos\phi_\omega + a^{(+)}_{2\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \\ - \; a^{(-)}_{1\omega j}\sinh\theta_{1\omega}\sin\phi_\omega - a^{(-)}_{2\omega j}\sinh\theta_{2\omega}\sin\phi_\omega \;, \end{aligned} \tag{4.3.15a}$$

$$\begin{aligned} a^{(+)}_{2\omega j}(\kappa) = -a^{(+)}_{2\omega j} \cosh\theta_{2\omega}\sin\phi_\omega + a^{(+)}_{1\omega j}\cosh\theta_{1\omega}\cos\phi_\omega \\ + \; a^{(-)}_{2\omega j}\sinh\theta_{2\omega}\sin\phi\omega - a^{(-)}_{1\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \;, \end{aligned} \tag{4.3.15b}$$

$$\begin{aligned} a^{(-)}_{1\omega j}(\kappa) = -a^{(+)}_{1\omega j} \sinh\theta_{1\omega}\cos\phi_\omega - a^{(+)}_{2\omega j}\sinh\theta_{2\omega}\sin\phi_\omega \\ + \; a^{(-)}_{1\omega j}\cosh\theta_{1\omega}\cos\phi_\omega + a^{(-)}_{2\omega j}\cosh\theta_{2\omega}\sin\phi_\omega \;, \end{aligned} \tag{4.3.15c}$$

$$\begin{aligned} a^{(-)}_{2\omega j}(\kappa) = -a^{(+)}_{2\omega j} \sinh\theta_{2\omega}\cos\phi_\omega + a^{(+)}_{1\omega j}\sinh\theta_{1\omega}\sin\phi_\omega \\ + \; a^{(+)}_{2\omega j}\cosh\theta_{1\omega}\cos\phi_\omega - a^{(+)}_{1\omega j}\cosh\theta_{1\omega}\sin\phi_\omega \;. \end{aligned} \tag{4.3.15d}$$

If we introduce the hermitian matrices

$$G = \prod_{i=1,2} \exp\sum_{\omega j}\left(\theta_{i\omega}(a^{(+)\dagger}_{i\omega j}a^{(-)\dagger}_{i\omega j} - a^{(+)}_{i\omega j}a^{(-)}_{i\omega j})\right) \;, \tag{4.3.16}$$

and

$$O = \exp(i\phi_\omega\Sigma_2) = \begin{pmatrix} \cos\phi_\omega & \sin\phi_\omega & 0 & 0 \\ -\sin\phi_\omega & \cos\phi_\omega & 0 & 0 \\ 0 & 0 & \cos\phi_\omega & \sin\phi_\omega \\ 0 & 0 & -\sin\phi_\omega & \cos\phi_\omega \end{pmatrix} , \tag{4.3.17}$$

where $\Sigma_2 = i\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then Eqs. (4.3.15a) – (4.3.15d) can be written in a compact form:

$$\begin{pmatrix} a^{(+)}_{1\omega j}(\kappa) \\ a^{(+)}_{2\omega j}(\kappa) \\ a^{(-)}_{1\omega j}(\kappa) \\ a^{(-)}_{2\omega j}(\kappa) \end{pmatrix} = O^{-1}G^{-1} \begin{pmatrix} a^{(+)}_{1\omega j} \\ a^{(+)}_{2\omega j} \\ a^{(-)}_{1\omega j} \\ a^{(-)}_{2\omega j} \end{pmatrix} GO \;. \tag{4.3.18}$$

Note that G is a product of two transformations (depending on the θ_i parameters), each of which mixes the creation and annihilation operators near

one black hole only, such that the corresponding expansion functions $H_{i\omega j}$ are analytic on the horizon of that black hole. The matrix O is a two dimensional rotation between the operators near different black holes. It depends on one parameter, ϕ_ω, to be chosen such that the modes be continuous in the region between the two black holes.

The vacuum states are given by

$$a^{(\pm)}_{i\omega j}\,|0\rangle = 0\ , \tag{4.3.19}$$

$$a^{(\pm)}_{i\omega j}(\kappa)\,|0(\kappa)\rangle = 0\ , \tag{4.3.20}$$

where

$$|0(\kappa)\rangle = O^{-1}G^{-1}\,|0\rangle\ .$$

The expectation value of the number of ωj modes is

$$\begin{aligned}\langle n_{\omega j}\rangle &= \langle 0(\kappa)\,|a^{(+)\dagger}_{1\omega j}a^{(+)}_{1\omega j} + a^{(+)\dagger}_{2\omega j}a^{(+)}_{2\omega j}\,|0(\kappa)\rangle \\ &= \sinh^2\theta_1 + \sinh^2\theta_2 = \frac{1}{e^{-\frac{2\pi\omega}{\kappa_1}}-1} + \frac{1}{e^{-\frac{2\pi\omega}{\kappa_2}}-1}\ .\end{aligned} \tag{4.3.21}$$

Note that our result does not depend on ϕ_ω, only on the $\theta_{i\omega}$ parameters. In other words, it depends only on how we match the solutions on the horizon.

Generalization to arbitrary N is straightforward. We divide the space into N cells, each of them containing only one black hole. We define normal modes which are nonvanishing in one cell only, and are given by Eqs. (4.3.2) and (4.3.3) except now i goes from 1 to N. A linear combination of these modes can be found that is analytic everywhere, by first matching the solutions on the horizons of the individual black holes, then in the region between the black holes, that is on the "walls" of the cells. Again, the corresponding Bogoliubov transformation has the form GO and the operators $a^{(\pm)}_{\omega j}$ can be thought of as

the components of a 2N component vector. G is a product of N transformations of the form given in Eq. (4.3.19) with i=1...N. The N parameters θ_i are chosen such that the new modes $H_{i\omega j}$ analytic on the horizon. Now O is an N dimensional rotation, mixing the particle states near the horizons of different black holes. The N(N $-$ 1)/2 parameters (ϕ_ω in the N=2 case) are to be chosen such that the solution is continuous in the region between the black holes. The expectation value of the ωj modes is again unaffected by these rotations, and, similar to Eq. (4.3.21), we obtain

$$\langle n_{\omega j}\rangle = \sum_{i=1}^{N} \sinh^2\theta_i = \sum_{i=1}^{N} \frac{1}{e^{-\frac{2\pi\omega}{\kappa_i}} - 1} \ . \tag{4.3.22}$$

The spectrum of the created particles is the sum of the thermal spectra of the individual black holes.

4.4 Neutrinos on the Schwarzschild Background

Now we examine massless spin one half fermions, that is neutrinos. As in the scalar case, we have to start with finding the normal mode expansion of the Dirac equation near the horizon. We will use the vierbein formalism. We shall see that, with a suitable choice of the vierbein fields, the Dirac equation is separable [32]. The metric tensor in this formalism is related to the flat metric $\eta_{\alpha\beta}$ through the vierbein V_μ^α, which satisfies orthonormality, $V_\mu^\alpha V_\beta^\nu = \delta_\beta^\alpha$, and completeness, $V_\mu^\alpha(x)V_\nu^\beta(x)\eta_{\alpha\beta} = g_{\mu\nu}$, conditions. In particular, in the Schwarzschild case the latter is

$$g_{\mu\nu} = V_\mu^\alpha(x)V_\nu^\beta\eta_{\alpha\beta} = \begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & -r & 0 & 0 \\ 0 & 0 & -r\sin\vartheta & 0 \\ 0 & 0 & 0 & -(1-\frac{2M}{r})^{-1} \end{pmatrix} , \tag{4.4.1}$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

and the indices α, β mean local frame indices and μ, ν are space-time indices. We can choose the vierbein such that its nonzero components are

$$
\begin{aligned}
V_t^0 &= (1-\frac{2M}{r})^{1/2}, & V_0^t &= (1-\frac{2M}{r})^{-1/2}, \\
V_\vartheta^1 &= r\ , & V_1^\vartheta &= \frac{1}{r}\ , \\
V_\phi^2 &= r\sin\vartheta\ , & V_2^\varphi &= \frac{1}{r\sin\vartheta}\ , \\
V_r^3 &= (1-\frac{2M}{r})^{-1/2}, & V_3^r &= (1-\frac{2M}{r})^{1/2}.
\end{aligned}
\tag{4.4.2}
$$

The massless Dirac equation in curved spacetime becomes

$$\gamma^\mu(\partial_\mu + \Gamma_\mu)\psi = 0\ , \tag{4.4.3}$$

where ψ is a Dirac spinor field with $(1-\gamma_5)\psi = 0$. The gamma matrices are given by

$$\gamma^\mu = V_\alpha^\mu \gamma^\alpha\ , \tag{4.4.4}$$

and they are the curved space counterparts of the usual flat space Dirac matrices, γ^α![1] They clearly satisfy the anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\ , \tag{4.4.5}$$

which are the curved space generalization of the flat space anticommutation relations,

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\ . \tag{4.4.6}$$

The spin connection, Γ_μ is given by

$$\Gamma_\mu(x) = \frac{1}{8}[\gamma^\alpha, \gamma^\beta] V_{\nu\alpha}(x) \nabla_\mu V_\beta^\nu(x)\ . \tag{4.4.7}$$

We are looking for solutions of the form

$$\psi = \begin{pmatrix} \eta \\ \eta \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = e^{-i\omega t} e^{-im\varphi} \begin{pmatrix} R_1(r)S_1(\vartheta) \\ R_2(r)S_2(\vartheta) \end{pmatrix}, \tag{4.4.8}$$

(1) Here we use the Dirac representation of the flat space Dirac matrices [33].

This leads to the following coupled first order differential equations for R_1, R_2, S_1, and S_2 :

$$
\begin{aligned}
r\sqrt{1-\frac{2M}{r}}\partial_r R_1 - \frac{i\omega r}{\sqrt{1-\frac{2M}{r}}}R_1 &= kR_2 \ , \\
r\sqrt{1-\frac{2M}{r}}\partial_r R_2 + \frac{i\omega r}{\sqrt{1-\frac{2M}{r}}}R_2 &= kR_1 \ , \\
\partial_\vartheta S_1 + \frac{m}{\sin\vartheta}S_1 &= kS_2 \ , \\
\partial_\vartheta S_2 - \frac{m}{\sin\vartheta}S_2 &= -kS_1 \ ,
\end{aligned}
\tag{4.4.9}
$$

where the separation constant k is to be chosen such that $S_1(\vartheta)$ and $S_2(\vartheta)$ are regular at $\vartheta = 0$ and $\vartheta = \pi$. We are interested in the solution of the radial equations at the past horizon. We find that

$$
R_1 \sim \exp(-i\omega u) \ , \tag{4.4.10}
$$

$$
R_2 \sim \exp(-i\omega v) \ . \tag{4.4.11}
$$

near the horizon. Thus, for neutrinos, the solution which corresponds to outgoing waves at the past horizon is

$$
\psi_\nu \sim e^{-im\varphi}e^{-i\omega u}S_1(\vartheta)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} . \tag{4.4.12}
$$

For antineutrinos the solution of the Dirac equation is given by charge conjugation ($C = i\gamma^2\gamma^0$):

$$
\psi_{\overline{\nu}} = C\overline{\psi}_\nu^T = \begin{pmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{pmatrix}\begin{pmatrix}\eta^*\\ \eta^*\end{pmatrix} = \begin{pmatrix}\eta_2^*\\ -\eta_1^*\\ -\eta_2^*\\ \eta_1^*\end{pmatrix} , \tag{4.4.13}
$$

where $\eta_1^* \sim \exp(i\omega u)$ represents outgoing negative frequency waves at the past horizon. Hence the solution corresponding to outgoing antineutrinos at the past horizon is given by

$$
\psi_{\overline{\nu}} \sim e^{im\varphi}e^{i\omega u}S_1^*\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} . \tag{4.4.14}
$$

To quantize we expand the field ψ

$$\psi = \sum_k \left(a_\omega u_\omega + b_\omega^\dagger \overline{v}_\omega + \text{h.c.} \right) , \tag{4.4.15}$$

where $a_\omega^\dagger$ and b_ω represent creation operators for neutrinos and antineutrinos respectively, while a_ω and $b_\omega^\dagger$ are the corresponding annihilation operators

$$a_\omega \, |0\rangle = b_\omega^\dagger \, |0\rangle = 0 \ ,$$

satisfying the following anticommutation relations:

$$\{a_\omega, a_{\omega'}^\dagger\} = \{b_\omega, b_{\omega'}^\dagger\} = \delta(\omega - \omega') \tag{4.4.16}$$

(the other anticommutators are zero). The spinors u_ω and $\overline{v}_\omega$ form a complete orthonormal set,

$$(u_\omega, u_{\omega'}) = (\overline{u}_\omega, \overline{u}_{\omega'}) = (v_\omega, v_{\omega'}) = (\overline{v}_\omega, \overline{v}_{\omega'}) = \delta(\omega - \omega') \tag{4.4.17}$$

(others are zero), and near the horizon they are given by

$$u_\omega = e^{-i\omega u} e^{-im\varphi} S_1(\vartheta) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sim e^{-\frac{i\omega}{\kappa} \ln|U|} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \tag{4.4.18}$$

$$\overline{v}_\omega = e^{i\omega u} e^{im\varphi} S_1^*(\vartheta) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \sim e^{\frac{i\omega}{\kappa} \ln|U|} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} . \tag{4.4.19}$$

Let us consider only neutrinos and define a complete set of positive frequency modes on the whole extended Schwarzschild spacetime:

$$F_{\omega j}^{(+)} = \begin{cases} e^{-\frac{i\omega}{\kappa} \ln|U|} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \text{outside the horizon} \\ 0 & \text{inside the horizon,} \end{cases} \tag{4.4.20}$$

$$F_{\omega j}^{(-)} = \begin{cases} e^{+\frac{i\omega}{\kappa} \ln|U|} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \text{inside the horizon} \\ 0 & \text{outside the horizon.} \end{cases} \tag{4.4.21}$$

The field ψ can be expanded in terms of complete set of modes $F^{(\pm)}_{\omega j}$ or $H^{(\pm)}_{\omega j}$:

$$\begin{aligned}\psi &= \sum_{\omega,j}\left(F^{(+)}_{\omega j}a^{(+)}_{\omega j} + F^{(-)}_{\omega j}a^{(-)\dagger}_{\omega j} + \text{h.c.}\right)\\ &= \sum_{\omega,j}\left(H^{(+)}_{\omega j}a^{(+)}_{\omega j}(\kappa) + H^{(-)}_{\omega j}a^{(-)\dagger}_{\omega j}(\kappa) + \text{h.c.}\right) ,\end{aligned} \tag{4.4.22}$$

where the modes $H^{(\pm)}_{\omega j}$ are defined by

$$H^{(+)}_{\omega j} = F^{(+)}_{\omega j}\cos\theta_\omega - F^{(-)}_{\omega j}\sin\theta_\omega , \tag{4.4.23}$$

$$H^{(-)}_{\omega j} = F^{(+)}_{\omega j}\sin\theta_\omega + F^{(-)}_{\omega j}\cos\theta_\omega . \tag{4.4.24}$$

Both $F^{(\pm)}_{\omega j}$ and $H^{(\pm)}_{\omega j}$ satisfy the orthonormality conditions

$$(F^{(\pm)}_{\omega j}, F^{(\pm)}_{\omega' j'}) = (H^{(\pm)}_{\omega j}, H^{(\pm)}_{\omega' j'}) = \delta_{jj'}\delta(\omega-\omega') . \tag{4.4.25}$$

As in the bosonic case, $H^{(\pm)}_{\omega j}$ are positive (negative) frequency modes, which are analytic on $\mathcal{H}^-$ if $\tan\theta_\omega = e^{-\frac{\pi\omega}{\kappa}}$. The corresponding creation and annihilation operators $a^{(\pm)}_{\omega j}(\kappa)$ are given by the Bogoliubov transformations:

$$\begin{aligned}a^{(+)}_{\omega j}(\kappa) &= \quad a^{(+)}_{\omega j}\cos\theta_\omega + a^{(-)\dagger}_{\omega j}\sin\theta_\omega\\ &= \exp(-G)a^{(+)}_{\omega j}\exp(G) ,\end{aligned} \tag{4.4.26}$$

$$\begin{aligned}a^{(-)}_{\omega j}(\kappa) &= -a^{(+)}_{\omega j}\sin\theta_\omega + a^{(-)\dagger}_{\omega j}\cos\theta_\omega\\ &= \exp(-G)a^{(-)}_{\omega j}\exp(G) ,\end{aligned} \tag{4.4.27}$$

where

$$G = \sum_{\omega j}\theta_\omega(a^{(+)\dagger}_{\omega j}a^{(-)\dagger}_{\omega j} - a^{(+)}_{\omega j}a^{(-)}_{\omega j}) . \tag{4.4.28}$$

The vacuum annihilated by $a^{(\pm)}_{\omega j}(\kappa)$ is given by

$$|0(\kappa)\rangle = \exp(-G)|0\rangle , \tag{4.4.29}$$

where $|0\rangle$ is the vacuum annihilated by $a^{(\pm)}_{\omega j}$.

The number of the ωj modes detected by an observer at infinity is given as a vacuum exceptation value

$$\langle n_{\omega j}\rangle_{\nu} = \langle 0(\kappa)|a^{(+)\dagger}_{\omega j} a^{(+)}_{\omega j}|0(\kappa)\rangle = \sin^2\theta_\omega = \frac{1}{e^{\frac{2\pi\omega}{\kappa}}+1} \,. \qquad (4.4.30)$$

For antineutrinos we have the same result,

$$\langle n_{\omega j}\rangle_{\overline{\nu}} = \sin^2\theta_\omega = \frac{1}{e^{\frac{2\pi\omega}{\kappa}}+1} \,. \qquad (4.4.31)$$

This is, as expected, a thermal distribution of fermions with effective temperature equal to the Hawking temperature.

The result obtained above can easily be extended to the case of many black holes. One should follow essentially the same steps as we have in Sect 4.3 for the bosonic case, and find similar results. In particular, one finds that the spectrum of the created fermions is the superposition of the thermal spectra of fermions created independently by the individual black holes.

Summarizing our results, we have found that, in agreement with previous calculations, the thermo-field approach lead to particle creation in a spacetime with causally disconnected regions. In the case of a single black hole the spectrum of the emitted particles is thermal with effective temperature equal to the Hawking temperature. In the case of well-separated black holes the spectrum is the superposition of individual black hole spectra. Furthermore our approach suggests that these results mainly depend on the analytic behavior of the fields on the horizon, but not on the statistics of the particles. This has been demonstrated by quantizing both a massless boson and a fermion field. Consequently, we hope that similar studies will lead to a better understanding of the properties of the horizon.

CHAPTER 5
CONCLUSIONS

We have investigated two extensions of Einstein gravity in 2+1 dimensions, Weyl gravity and topologically massive gravity. We have also considerated the applications of thermo-field dynamics to particle creation by black holes.

In the case of Weyl gravity we considered the consequences of duality in the context of Weyl theory in three dimensions. We constructed a theory of gravity with Weyl invariance and a noncanonical scalar auxiliary field, as a laboratory to study duality between the gauge field and its field strength. There it appears as an equation of motion. We have studied the classical solutions, and found that they can be classified by the nonvanishing components of the field strength. There are stationary solutions only if the electric field is vanishing. If the magnetic field is vanishing as well, *i.e.*, in the pure gauge case, our theory reduces to Einstein gravity in flat or de Sitter space. The general solution was found. In the case when only the magnetic field is nonvanishing, the problem reduces to the solution of a Liouville equation. We studied the axial symmetric solutions in more detail. We found that the solutions have the helical-conical structure, characteristic to 2+1 dimensional gravity, also they are 2+1 dimensional analogs of the known 3+1 dimensional Gödel and Taub-NUT type solutions. Interestingly, the "matter part" was described by a rotating Chern-Simons fluid with intriguing properties. Consequently this work might have interesting applications in fluid mechanics.

Next we studied the renormalizability of TMG by using nonlocal regularization. We found that the theory is renormalizable under a certain assumption, namely when the nonlocal measure factor exists. Although we cannot give a general proof of its existence, it can be constructed perturbatively, and its existence and gauge invariance can be checked to any order. We showed that a possible anomaly which could spoil its power counting renormalizability does not occur. If our assumption is valid, topologically massive gravity is the only known example of a renormalizable and dynamical theory of gravity.

Finally, we have used thermo-field dynamics to study particle creation in causally disconnected spacetimes. We have chosen 3+1 dimensional black hole spacetimes, because these are the best known examples with the above property. We have found that our results are consistent with those obtained by different methods. In particular the thermal character of the vacuum has been derived for the emission of massless scalar particles and for the emission of neutrinos. We also discussed how to generalize the method to space times with many disconnected regions. For definiteness we considered the example of many well-separated black holes, and found that the spectrum is the superposition of the individual black hole thermal spectra. Approximations were necessary in order to obtain the multi-black hole metric, but only because we do not know any exact solutions of the Einstein equation with the above property, not because of the failure of our formalism in a more accurate case. At the same time our example clearly shows the basic ideas.

This method not only provides a new technical tool to discuss the quantization in such spacetimes, but it also helps us to understand the features of the particle creation process by looking at the problem from a new point of view. In particular, the role of the event horizon has a new interpretation. Namely,

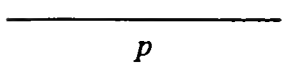
we have found that the analytic behavior of fields on event horizons is crucial to the derivation of the spectrum of the created particles. The effect does not depend on the statistics of the particles. This has been demonstrated explicitly by quantizing a massless scalar field and a neutrino field on the Schwarzschild background using thermo-field dynamics. In both cases the spectrum of the produced particles is thermal with effective temperature equal to the Hawking temperature.

As in earlier works, we also have found that the particle creation process is due to the presence of the event horizon. To see this one should note that the physical observables are the (temperature dependent) vacuum expectation values, and they contain information only about particle states outside the horizon(s) (only these states have nonzero contribution). But we have learned more than that by realizing that the spectrum of the radiation depends on the properties of the event horizon, namely on the number of disconnected pieces, and on the behavior of the fields near the horizon.

APPENDIX A
FEYNMAN RULES FOR TMG

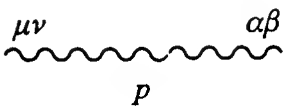
The Feynman rules for TMG follow from the action described in Eqs. (3.3.1), (3.3.7), (3.3.8) and (3.3.9) in the usual way. The corresponding rules of the nonlocalized theory can be obtained by applying the general rules of Sect. 3.5 to TMG. Associated with each field of the local theory is an auxiliary field. Thus in the nonlocal theory auxiliary fields Ψ, $k_{\mu\nu}$ and d^{μ} are associated with the original fields Φ, $h_{\mu\nu}$ and c^{μ}. The ghost field b_{μ} does not require an auxiliary field because its BRS variation is linear. In the nonlocalized theory the form of the vertices are unchanged except that now the lines represent both physical and auxiliary lines. The propagators of the local theory are replaced by smeared ones in the nonlocal theory. The original propagators are multiplied by $\mathcal{E}_0^2$, defined in (3.6.1), while the auxiliary field propagators are $1-\mathcal{E}_0^2$ multiplied by the original ones. In constructing Feynman diagrams in the nonlocal theory, one does not include loops involving only auxiliary fields or diagrams with auxiliary fields on external lines. Details can be found in Sects. 3.4 and 3.5.

Below we list the Feynman rules for TMG where the rules for the nonlocal theory are described above. An auxiliary field line is represented by putting a bar on the corresponding physical field line. First we list the propagators. The Φ propagator is



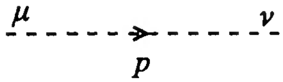
$$\frac{i}{16p^2} .$$

The h propagator is



$$\frac{1}{4p^4}p_\gamma(\epsilon^{\mu\alpha\gamma}P^{\nu\beta}+\epsilon^{\nu\alpha\gamma}P^{\mu\beta}+\epsilon^{\mu\beta\gamma}P^{\nu\alpha}+\epsilon^{\nu\beta\gamma}P^{\mu\alpha})\ .$$

Finally, the ghost propagator is

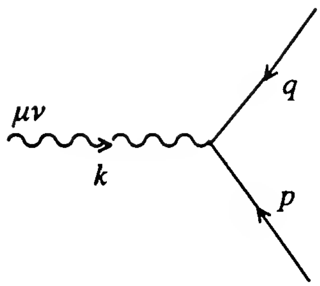


$$\frac{i}{p^2}(\eta_{\mu\nu}-\frac{1}{4p^2}p_\mu p_\nu)\ .$$

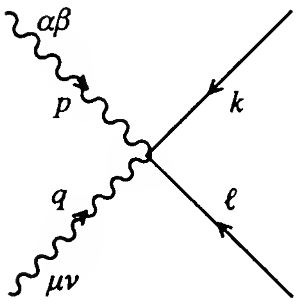
where $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1)$, $\epsilon_{\alpha\beta\gamma}$ is the 2+1 dimensional Levi-Cività tensor and the projection operator $P^{\mu\nu}$ is

$$P^{\mu\nu}=\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\ . \tag{A.1}$$

Next we give the vertices. Since the vertices involve all orders in the h field we shall only give vertices at most cubic in the h field and no higher than 4 point vertices. The vertices arising from the Einstein and kinetic terms of (3.3.7) are quadratic in the Φ field and contain two derivatives. The lowest order vertices are

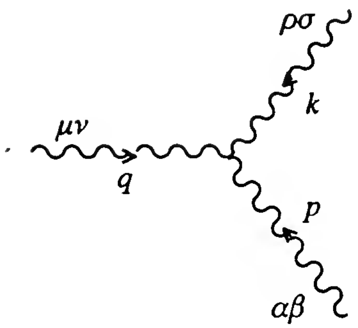


$$16i\sqrt{\mu}p_\mu q_\nu\ ,$$



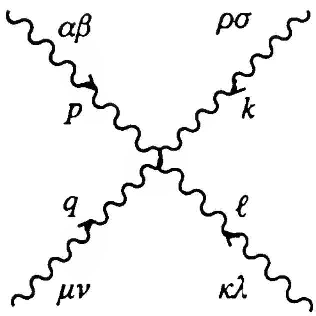
$$i\mu\eta_{\alpha\mu}\{\eta_{\beta\nu}(8k\cdot\ell+3p\cdot q+2(p^2+q^2))-16(k_\nu\ell_\beta+k_\beta\ell_\nu)-2p_\nu q_\beta\}\ .$$

The vertices from the Chern-Simons term contain only h fields and three derivatives. The lowest order vertices coming from this term are



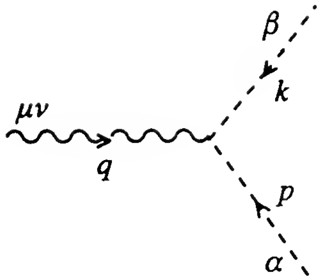
$$
\begin{aligned}
&-\tfrac{1}{4}\sqrt{\mu}\,\epsilon^{\gamma\kappa\tau}(-p_\gamma k_\kappa q_\rho \eta_{\beta\sigma}\eta_{\alpha\mu}\eta_{\tau\nu} \\
&+ p_\rho k_\mu k_\kappa \eta_{\nu\alpha}\eta_{\gamma\beta}\eta_{\tau\sigma} - p_\nu k_\mu k_\kappa \eta_{\rho\alpha}\eta_{\gamma\beta}\eta_{\tau\sigma} \\
&+ p_\nu k_\alpha k_\kappa \eta_{\mu\rho}\eta_{\gamma\beta}\eta_{\tau\sigma} - p\cdot k\, k_\kappa \eta_{\mu\alpha}\eta_{\nu\rho}\eta_{\gamma\beta}\eta_{\tau\sigma} \\
&+ \tfrac{1}{3} p_\mu k_\alpha q_\rho \eta_{\gamma\beta}\eta_{\kappa\sigma}\eta_{\tau\nu} + p\cdot q\, k_\mu \eta_{\alpha\rho}\eta_{\gamma\beta}\eta_{\kappa\sigma}\eta_{\tau\nu}) \\
&+ \text{permutations,}
\end{aligned}
$$

and

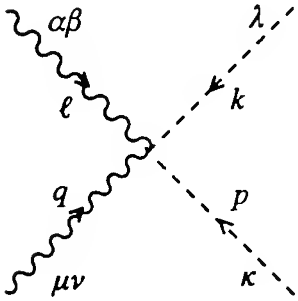


$$
\begin{aligned}
&-\tfrac{1}{24}\mu\Big\{\epsilon^{\lambda\nu\sigma}\Big(\eta_{\alpha\kappa}\eta_{\beta\mu}(-k\cdot\ell\, q_\rho + k\cdot q\,\ell_\rho) \\
&\qquad + 2\eta_{\alpha\kappa} q_\beta (k\cdot\ell\,\eta_{\mu\rho} - k_\mu \ell_\rho) \\
&\qquad + \ell_\alpha q_\beta (k_\mu \eta_{\kappa\rho} - k_\kappa \eta_{\mu\rho})\Big) \\
&+ \epsilon^{\lambda\gamma\nu}\Big(-2k_\gamma(\eta_{\alpha\mu}\eta_{\beta\kappa} q_\sigma \ell_\rho - \eta_{\rho\mu}\eta_{\kappa\sigma} q_\alpha \ell_\beta) \\
&\qquad + 3\ell_\gamma(\eta_{\beta\sigma} q_\rho(\ell_\mu \eta_{\alpha\kappa} - \ell_\alpha \eta_{\kappa\mu}) \\
&\qquad + \eta_{\beta\rho}\eta_{\mu\sigma}(\ell_\alpha q_\kappa - \ell\cdot q\,\eta_{\alpha\kappa}) \\
&\qquad - 2\eta_{\alpha\kappa}\ell_\rho(q_\sigma \eta_{\beta\mu} - q_\beta \eta_{\mu\sigma}))\Big) \\
&+ \epsilon^{\lambda\gamma\delta} q_\delta\Big(k_\gamma \eta_{\alpha\rho}(\eta_{\beta\mu}(\eta_{\kappa\nu}\ell_\sigma - \eta_{\kappa\sigma}\ell_\nu) \\
&\qquad + 4\eta_{\mu\sigma}(\eta_{\beta\kappa}\ell_\nu - \eta_{\kappa\nu}\ell_\beta)) \\
&\qquad + 3\ell_\gamma \eta_{\beta\rho}\eta_{\nu\sigma}(\ell_\alpha \eta_{\kappa\mu} - \ell_\mu \eta_{\alpha\kappa})\Big) \\
&+ 2\epsilon^{\gamma\delta\epsilon}\ell_\gamma q_\delta k_\epsilon \eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\kappa\sigma}\eta_{\lambda\nu}\Big\} \\
&+ \text{permutations.}
\end{aligned}
$$

The vertices arising from the ghost part of the action in (3.3.9) are



$$-i\sqrt{\mu}(p_\mu k_\nu \eta_{\alpha\beta} + p_\beta k_\mu \eta_{\nu\alpha} - p_\mu q_\beta \eta_{\alpha\nu} - \tfrac{2}{3} p_\mu k_\beta \eta_{\nu\alpha} - \tfrac{2}{3} p_\alpha k_\mu \eta_{\nu\beta}) \ ,$$

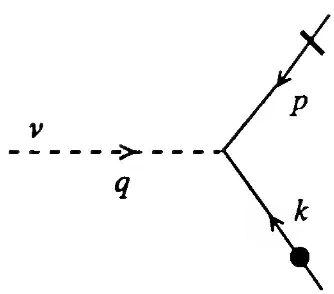


$$\tfrac{2}{3} i\mu (p_\mu k_\beta \eta_{\alpha\lambda}\eta_{\kappa\nu} + p_\beta k_\mu \eta_{\alpha\kappa}\eta_{\lambda\nu}) \ .$$

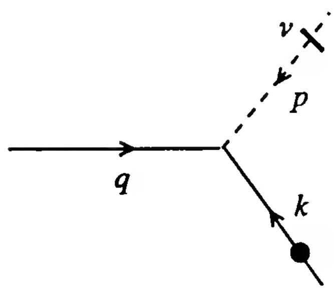
Here it is understood that one must symmetrize each pair of indices on an external h line and take permutations of the various identical h fields arising from a vertex as indicated in the figures.

APPENDIX B
MEASURE FACTOR
FEYNMAN RULES FOR TMG

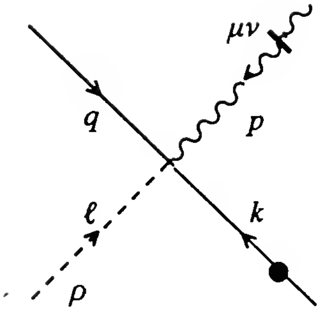
The Feynman rules for the nonlocal BRST variation of the measure were discussed in Sect. 3.5 and were derived from Eq. (3.5.4). There are two types of vertices, those from the nonlocal theory as decribed in Appendix A and the other type connects to internal lines, one of which has "propagator" $\mathcal{E}_0^2$. Such internal lines are represented with a dot on their legs. All the diagrams contibuting to the variation of the measure factor are one loop diagrams and contain only one of the second type of vertex and an arbitrary number of vertices from the original theory. All dashed lines correspond to a ghost field c^μ in the vertices that follow. Those diagrams coming from the contribution of the variation of the Φ field are



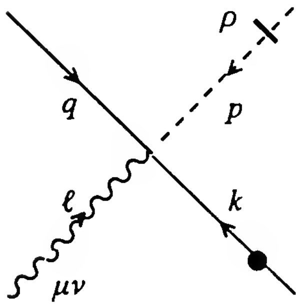
$$-\tfrac{1}{6}q_\nu + p_\nu \ ,$$



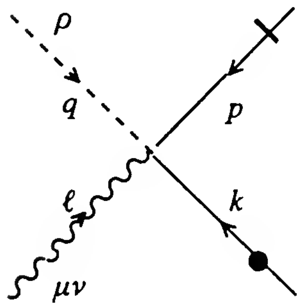
$$-\tfrac{1}{6}p_\nu + q_\nu \ ,$$



$$-\tfrac{1}{12}\sqrt{\mu}(\eta_{\mu\rho}q_\nu + \eta_{\nu\rho}q_\mu) \ ,$$

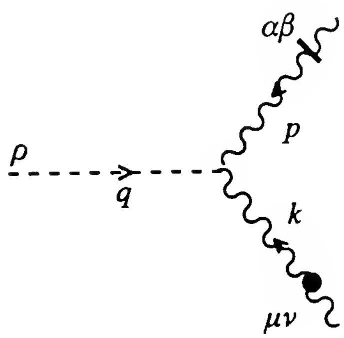


$$\tfrac{1}{12}\sqrt{\mu}(\eta_{\mu\rho}p_\nu + \eta_{\nu\rho}p_\mu) \ ,$$

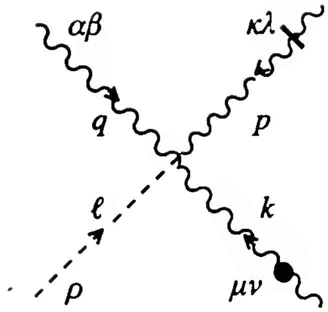


$$-\tfrac{1}{12}\sqrt{\mu}(\eta_{\mu\rho}q_\nu + \eta_{\nu\rho}q_\mu) \ .$$

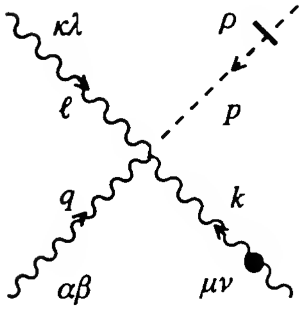
The vertices arising from the contribution of the variation of h are



$$-\sqrt{\mu}\{q_\alpha(\eta^\mu_\beta\eta^\nu_\rho + \eta^\nu_\beta\eta^\mu_\rho - \tfrac{2}{3}\eta^{\mu\nu}\eta_{\rho\beta}) \\ - \eta^\mu_\beta\eta^\nu_\alpha(p_\rho + \tfrac{2}{3}q_\rho)\} \ ,$$

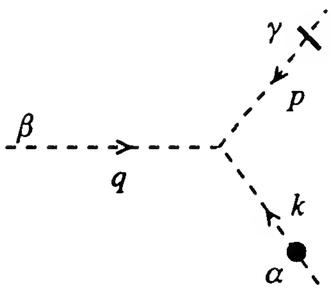


$$\tfrac{2}{3}\sqrt{\mu}(\ell_\beta \eta_{\alpha\rho}\eta^\mu_\kappa \eta^\nu_\lambda + \ell_\lambda \eta_{\kappa\rho}\eta^\mu_\alpha \eta^\nu_\beta) ,$$



$$\tfrac{2}{3}\sqrt{\mu}(p_\beta \eta_{\alpha\rho}\eta^\mu_\kappa \eta^\nu_\lambda + p_\lambda \eta_{\kappa\rho}\eta^\mu_\alpha \eta^\nu_\beta) .$$

Finally, the vertex coming from the contribution of the variation of the ghost field is



$$-p_\beta \eta_{\alpha\gamma} + q_\gamma \eta_{\alpha\beta} .$$

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BIOGRAPHICAL SKETCH

Bettina E. Keszthelyi was born April, 8, 1961, in Budapest, Hungary. She received the Diploma in Physics from Eötvös University in 1984. She then travelled to the United States to study in the graduate program in astrophysics at the University of Chicago from 1984 to 1986. She decided to change her research direction to the area of particle physics and subsequently went to the University of Florida from 1987 to 1993. She spent much of the 1992 to 1993 academic year on leave studying field theory at Brandeis University. She is currently an Honorary Fellow at the University of Wisconsin.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Pierre M. Ramond, Chairman
Professor of Physics

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Pierre Sikivie
Professor of Physics

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James R. Ipser
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James N. Fry
Associate Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Louis S. Block
Professor of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Physics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 1993

Dean, Graduate School